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PRACTICAL
DESCRIPTIVE GEOMETRY

Published by the

McGraw-Hill Book Company

New York

Successors to the Book Departments of the

McGraw Publishing Company — Hill Publishing Company

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Electric Railway Journal
Metallurgical and Chemical Engineering
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PRACTICAL DESCRIPTIVE GEOMETRY

BY

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GEORGE E. MAYCOCK

McGRAW-HILL BOOK COMPANY

239 WEST 39TH STREET, NEW YORK

6 BOUVERIE STREET, LONDON, E. C.

1912

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PREFACE

In presenting a new text-book on Descriptive Geometry, the writer is fully aware of the excellence of existing treatises, and appreciates what they have contributed toward a higher standard of technical training. The authors of these treatises, pursuing their several courses, have carried the development of the subject to a high plane.

However, in spite of the indisputable excellence of many of the text-books, they seem to have failed to arouse the interest of the student, partly by ignoring the practical applications, and partly by making only a slight attempt to present the subject attractively. Some of the books are valuable only for reference and are useless in the class room; others are incomplete in the essentials; some are faddish, emphasizing certain features and treating the rest inadequately, while even the best convey to the student only a very slight idea of the practical value of the subject. This has given rise to the belief, prevailing almost universally in the student body, that Descriptive Geometry is merely a disciplinary study, having little or no relation to the life work of the individual.

The aim of the writer has been, therefore, to present the subject to the student in a simple manner, as progressively as possible, reminding him constantly of the relation which exists between Descriptive Geometry and Practical Drafting; and to avoid needless difficulty by using language and directions of the greatest possible clarity. The writer believes that a thorough knowledge of the subject is achieved not through much study of the text, but by working exercises. To this end he has provided a large number of exercises, scattered through the text, of considerable variety and capable of infinite multiplication by the clever instructor.

An examination of the subject will reveal the following features and innovations:

(1) A thorough drill in fundamentals.

(2) Repetitions of statements for the sake of emphasis.

(3) Notation comprehensive, yet reduced to its lowest terms.

(4) Analyses separated from proofs.

(5) Tabulated order of the operations in each analysis and construction.

(6) Exercises to the number of 860, of which about one-fourth are such as may be met in actual practice. Most of these are dimensioned for a space of standard size.

(7) Notes of exceptions, checks on solutions, special cases, hints, and rules presented wherever they seem necessary or advisable.

(8) Brief treatment only of such special subjects as Shades and Shadows, Perspective, etc.

(9) The original seventeen "Point, Line, and Plane" problems expanded to forty-three.

(10) New treatment of surfaces, less importance being attached to passing tangent planes, and more importance to plane sections, intersections, developments, and practical applications. For example, there are about thirty exercises which are based on practical illustrations of the various warped surfaces, showing actual uses for all the listed varieties.

The expansion of the problems noted in feature (9) is justified, the writer believes, in order to fully cover the ground, and to provide the necessary steps in the development of the subject. Some of them are special cases and some require little or no change in the analysis from those of preceding problems.

Students undertaking the study of Descriptive Geometry should have a fair knowledge of Mechanical Drawing, at least as much as is usually included in High School courses. The minimum prerequisite should amount to the work contained in the first eight chapters of French's "Engineering Drawing," or in Reid's "Mechanical Drawing" entire.

In preparing this text-book the writer has consulted most of the standard works on the subject, and desires to make special acknowledgement for ideas and information to the excellent books by MacCord, Church and Bartlett, Phillips and Millar, and Randall. Most of the exercises are original, but many have been adapted from such admirable books of exercises as those by Professors Hood, Marshall, and Fishleigh, as well as the problem files of Armour Institute of Technology, and the books of the American School of Correspondence.

The writer wishes to thank his colleagues, Robert V. Perry, Henry L. Nachman, John S. Reid, and Charles R. Swineford, of the Faculty of Armour Institute of Technology, for valuable suggestions and encouragement.

William Griswold Smith.

Armour Institute of Technology,
Chicago, June 20, 1912.

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PRACTICAL DESCRIPTIVE GEOMETRY

CHAPTER I

DEFINITIONS, NOTATION, PRELIMINARY THEOREMS, AND EXERCISES.

1. Definition.—**Descriptive Geometry** is the Science of **Mechanical Drawing.**

Mechanical Drawing is the **Art** of representing objects on paper, or other flat surface, by means of **projections** of the objects on imaginary planes.

Projections are images of objects made by imaginary lines projected from the objects, and are variously made for various uses, and may be thus classified:

Orthographic Projection: objects projected by perpendicular lines.

Perspective or **Scenographic Projection:** objects projected by converging lines.

Oblique Projection: objects projected by parallel oblique lines.

Isometric Projection, Cavalier Projection, and **Pseudo Perspective** are variants on **Perspective,** using Orthographic or Oblique Projection. They are termed collectively **Single Plane Projection,** and treated in a separate chapter.

Orthographic Projection

2. Discussion.—This is by far the most important class of projection, and is almost universally used in making a "working drawing" and in "machine sketching." To thoroughly understand its significance, let us first consider the difference between a picture and a working drawing.

A picture, or perspective, is a representation of an object, group, or scene, as it appears to the eye. Fig. 1 shows the way a picture is projected. The lines of light from any object reach the eye in converging lines. In this figure, A is the point of sight, B and C are objects of equal size, but at different distances from A, and D is a plane of projection between the eye and the

objects. Converging rays (shown by the dotted lines), from the tips of B and C to the eye, pierce the plane D at certain points. These points determine the size of the pictures of B and C, and the images B′ and C′ are these pictures. As B is the nearer, B′ is the larger. Thus distant objects appear smaller than nearer ones of equal, and often smaller size. By means of this fact, and the knowledge of the beholder of the comparative

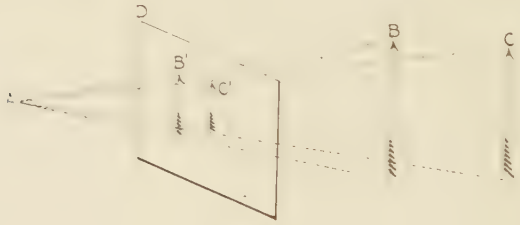


Fig. 1.

sizes of the objects, a picture shows the entire composition of a scene.

The working drawing takes no such account of distances. Projections are made by rays perpendicular to the plane, and therefore parallel to each other, which means that equal-sized objects, whether near or remote, will have equal-sized projections. Fig. 2 shows this fact. In orthographic projection the eye is

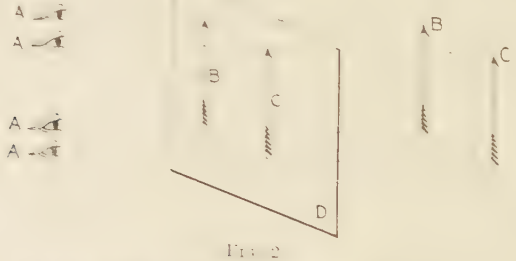


Fig. 2

considered to be at an infinite distance. If the effect of a finite position of the eye is desired, the eye must be shifted (as A in Fig. 2) to the line of each ray. This, of course, is an artificial condition, and proves that the working drawing is not, and cannot be, a correct visual image of the object.

This fact would seem to be a disadvantage against the use of orthographic projection, but it is actually much more useful in

manufacture and construction than the perspective, and much easier for the draftsman to use. On account of the complexity of many machines and structures, many dimensions are necessary in many directions, interior dimensions, thicknesses, and irregularities have to be shown, which are all but impossible in a picture. This is accomplished by projecting the object orthographically on two or more planes, called the **Planes of Projection.**

The Planes of Projection

3. The two principal planes are the **Vertical** and **Horizontal.** The Vertical Plane (which we shall hereafter write V) is a plane normal to the earth's surface; that is, perpendicular to the apparent surface of the earth. The Horizontal Plane (hereafter written H) is a plane perpendicular to any vertical line; that is, it is parallel to the apparent surface of the earth. H and V are perpendicular to each other. The projections on these two planes determine the relative position of any two or more points in space. There are conditions arising often, which render a third plane desirable, and indeed necessary, in the solution of certain problems. This third plane is perpendicular to both H and V, and is called the **Profile Plane,** which will always be written P.

Any number of **auxiliary planes,** which may be parallel, perpendicular, or oblique to H or V, may be used, according to the complexities of the problem. These are commonly called Q in this book. (In the architect's plans of a big building, two different vertical projections may be needed, two profiles, a horizontal projection for every floor, basement, and roof, auxiliary projections of certain oblique features, sectional interior projections, and a hundred "detail" projections of doors, windows, staircases, elevators, etc., all in orthographic projection. Besides these, it is customary to make a perspective of the projected building with colors, trees, grass, walks, etc. The average purchaser cannot comprehend working drawings, while the picture is of little use to the workman. Perspectives are for the customers, orthographic projections for the workmen.)

Representation on the Flat

4. It is obvious that it would be awkward and impracticable to have a drawing board made with horizontal, vertical, and

profile sides, and the respective projections made on them. Not only would it be inconvenient, but it is entirely unnecessary. Fig. 3 shows H and V intersecting, as they must, in a line, called the Ground Line (always designated GL). By revolving V about GL as an axis, or hinge, it will be brought into coincidence with H, and H and V projections can be drawn on one surface. Similarly, P may be revolved into the same surface about its intersection with V or H. (For mere solution of problems it is

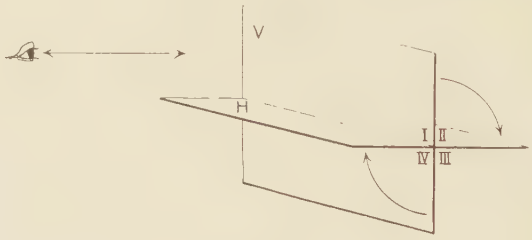


Fig. 3.

immaterial whether P is revolved about its intersection, or **trace,** with V or with H, but in practical work it is far more often, if not always, revolved about the V-trace.) In this work the profile will always be represented as rotated about its V-trace.

Note.—When working at the blackboard, it will be easier to regard the foregoing rotation as being made thus: rotate H into coincidence with V, by dropping the front portion of H, and raising the part behind V. Whichever way it is considered, the First and Third angles are opened, and the Second and Fourth Angles are closed.

The Four Dihedral Angles

5. It will be seen from Fig. 3 that the planes of projection do not end at GL. In fact, they are unlimited in extent, and divide space into four dihedral angles, with GL as their common intersection. In folding into the flat, therefore, no portion of the drawing board is monopolized by either H or V. There is a portion of H above GL, and a portion below GL. There is a portion of V above GL, and a portion below GL. (This circumstance will be found essential to the solution of certain problems of abstract character. In practical work the different projections, or **views,** as they are often called, are entirely separated from each other, and definite spaces are reserved for H, V and P.)

The four dihedral angles, in any of which objects may be imagined, are numbered, as shown by the Roman numerals in Fig. 3.

First Angle, in front of *V*, above *H;*
Second Angle, behind *V*, above *H;*
Third Angle, behind *V*, below *H;*
Fourth Angle, in front of *V*, below *H*.

The first angle (designated in this work as *I*) is chiefly used in our problems, on account of its accessibility and the ease of presentation by the teacher to the class, when giving a solution of a problem in space.

The third angle (*III*) is commonly used in practical work, for the reason that the *H*-projection is usually made **looking down** on the object, and besides for this angle the profile comes out **right-handed** and detached, whereas the usual method of swinging a profile from a first-angle-projection will either bring the object out left-handed, or mixed up with the other views. Many of our practical problems are given in *III*.

6. Commercial Terminology of Projections

As the terms *V*-projection, *H*-projections, etc., are not descriptive to non-professionals, certain terms are used by draftsmen.

The *V*-projection is often called the **Elevation, Front View, Rear View,** or **Sectional Elevation,** as the case may be.

The *H*-projection is called the **Plan, Top View, Bottom View,** or **Sectional Plan,** as the case may be.

The *P*-projection is called the **Profile, End View, Right** or **Left End View,** or **Profile Section,** as the case may be.

The Representation of Points, Lines, and Planes

7. Points.—It is obvious that the projection of a point on one plane does not determine that point. For example, let a point on the floor be the projection of some point in the room. How high is that point? It may be any distance. The point may be the projection itself, or the point may be on the ceiling above, or anywhere in between. To locate the point in space by one projection, it is necessary to add its distance, positive or negative, from the projection. A better way (the draftsman's way) is to locate the point graphically. By projecting that point on the wall, its height is determined and shown.

This definitely locates a point, for, if perpendicular lines be erected from each projection of a point, these perpendiculars will pass through the point. As there can be but one intersection

of the perpendiculars, the point is absolutely located in space by its projection on two planes. The perpendicular lines from the point to the planes of projection are called **projectors.**

8. The H- and V-projections of a point must lie on a common perpendicular to GL.

For the H-projector and the V-projector are both perpendicular to the respective planes of projection, thus determining a plane perpendicular to both H and V, therefore perpendicular to their intersection, GL. This plane being perpendicular to GL, its intersections with H and V will both be perpendicular to GL, and **at the same point,** which proves the statement. The P-projection is obtained by the intersection of the P-projectors from the H- and V-projections of the point, when P is revolved into V.

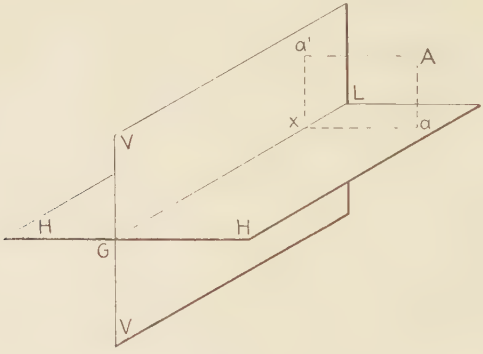


FIG. 4.

9. In Fig. 4 the point A in space is projected on V in the point a′, and on H in the point a. The projector Aa′ is parallel and equal to ax (its projection on H), because Aa′ is parallel to H, being perpendicular to V. Likewise Aa is parallel and equal to a′x. This fact proves the following theorem.

Theorem I.—The distance that any point is from H is equal to the distance from its V-projection to GL; the distance that any point is from V is equal to the distance from its H-projection to GL.

(**Note.**—The theorems in this chapter can all be proved by Plane and Solid Geometry, and form the basis of the problems that follow. The student should be familiar with them and their proof.)

PROJECTIONS OF POINTS IN THE VARIOUS ANGLES

10. From Fig. 4 it may be seen that, in folding the planes into the flat, a′ will be above *GL*,and a will be below *GL*.

This relation between the projections only obtains in angle *I*. Fig. 5 gives a pictorial representation of points in all the angles, and their revolution into the flat. A, in *I*, shows its *H*-projection rotated below *GL*. B, in *II*, shows its *H*-projection rotated

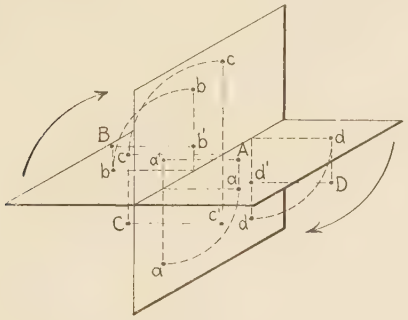


Fig. 5.

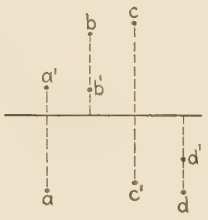


Fig. 5a.

above *GL*. *C*, in *III*, shows its *H*-projection rotated above *GL*. D, in *IV*, shows its *H*-projection rotated below *GL*. Fig. 5*a* shows the flat representation of *A*, *B*, *C*, and *D*.

Theorem II.—Points located in front of V have their H-projections below GL. Points behind V have their H-projections above GL. Points above H have their V-projections above GL. Points below H have their V-projections below GL.

Theorem III.—Points in H have their V-projections in GL, and points in V have their H-projections in GL.

Proof.—A point in *H* is zero distance from *H*, hence its *V*-projection is zero distance from *GL*. (Theorem I.)

11. Designation.—The designation of an **actual point** in space is a capital letter, while the **projections** of the point are designated by small (lower case) letters. The *H*-projection of a point is distinguished from the *V*-projection of the same point by the use of the "prime" in connection with the *V*-projection. Thus, the point A will have its *H*-projection a, and its *V*-projection a′. For the *P*-projection we use a superscript (^{P}), a^{P}.

Revolved Points.—When points are moved from their original position by revolution, or otherwise, the new position is written with a subscript; thus, a_1 means that A is in a revolved position and this is its new horizontal projection. The second new position would be written a_{11} and a'_{11} for the two projections, or if it were the P-projection, a^P_{11}, etc.

12. Notation.—In giving out a point for a problem, we write; (1) its distance to the right, measured along GL, (2) the distance above or below GL of its V-projection, and (3) the distance above or below GL of its H-projection. Fig. 6 shows the location on

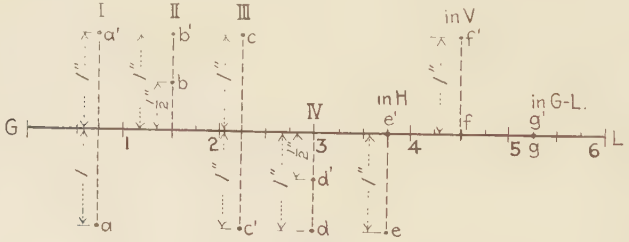


Fig. 6.

paper of the following points, using the notation that is used throughout the work: A($\frac{3}{4}+1-1$), B($1\frac{1}{2}+1+\frac{1}{2}$), C($2\frac{1}{4}-1+1$), D($3-\frac{1}{2}-1$), E($3\frac{3}{4}$, $0-1$), F($4\frac{1}{2}+1$, 0), G($5\frac{1}{4}$, 0, 0).

Example.—The point A is located $\frac{3}{4}$ in. to the right, with its V-projection 1 in. above GL, and the H-projection 1 in. below GL. This reads A($\frac{3}{4}+1-1$). Plus always means **above** GL, and minus always means **below** GL, and the V-projection is always given before the H-projection.

Note.—The unit in this notation may be 1 inch for drawing-board work, and 5 inches for blackboard. Referring to Fig. 6, it is not necessary to put in the dimensions shown there, but they are given the student, so that he may compare their distances with the figures in the notation. As no work in Descriptive Geometry can be done without notation, the student must have a thorough drill in the following exercises:

Exercises in Point Representation

Hints.—In laying out any problem or exercise, do not try to locate the points in space. No draftsman does that, he does his work on paper. Locate the points on the paper, remembering that plus means above, and minus means below

GL, and that the first amount means the distance to the right, the second the *V*-projection, and the third the *H*-projection. The Roman numerals that appear in Fig. 6 refer to the space angle in which the point is located.

LAYOUT FOR DRAWING PLATES

13. All exercises in this work are intended to fit in a space 6 in. × 7 in., except in a few cases, where dimensions are specified. Fig. 7 shows an excellent layout for drawing plates. A sheet

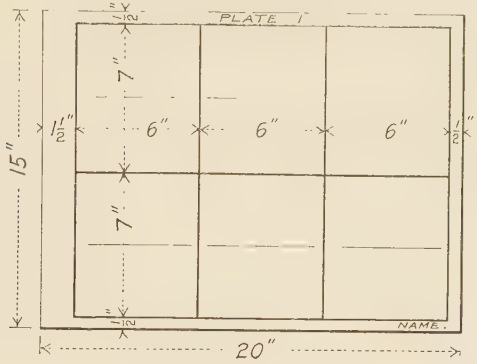


Fig. 7.

15 in. × 20 in. is given a 1½-in. margin on the left for binding, and a ½-in. margin on the other three sides. This gives a working space 14 in. × 18 in., divided into six rectangles, each 6 in. × 7 in., with *GL* through the middle of each.

Inking Plates.—It is not advisable to ink in plates, except possibly the border lines, ground lines, and lettering. Good pencil work, and neat, legible lettering should be insisted on.

Projectors.—Make the lines running from the projections to *GL*, called *projectors*, light, dotted lines. See Fig. 5*a* and Fig. 6.

Letter all points.

EXERCISES

14. Locate on the flat and designate in space (see the Roman Numerals, etc., in Fig. 6) the following points:

1. A($\frac{3}{4}+2-2$), B($1\frac{1}{2}-1+2\frac{1}{2}$), C($2\frac{1}{4}-2$, 0), D(3, 0, 0), E($3\frac{3}{4}-2+1$), F($4\frac{1}{2}+1+1$), G($5\frac{1}{4}$, $0-\frac{1}{2}$).
2. K($\frac{3}{4}-1-1$), L($1\frac{1}{2}$, $0+1$), M($2\frac{1}{4}$, $0-2$), N($3+1\frac{1}{2}$, 0), O($3\frac{3}{4}-2$, 0), P($4\frac{1}{2}+1+1$), Q($5\frac{1}{4}+\frac{1}{2}-1$).

3. A($\frac{3}{4}$, 1 before V, 2 below H), B($1\frac{1}{2}$, $2\frac{1}{4}$ above H, 1 before V), C($2\frac{1}{4}$, 1 below H, 1 behind V), D(3, $1\frac{1}{2}$ behind V, $1\frac{1}{2}$ above H), E($3\frac{3}{4}$, in H, 2 behind V), F($4\frac{1}{2}$, before V, in H), G ($5\frac{1}{4}$, in V, 1 below H).
4. K($\frac{3}{4}$, in GL), L($1\frac{1}{2}$, $1\frac{1}{4}$ behind V, $1\frac{1}{4}$ above H), M($2\frac{1}{4}$, in V, $1\frac{1}{2}$ above H), N(3, $\frac{3}{4}$ below H, 1 behind V), O($3\frac{3}{4}$, $\frac{1}{2}$ before V, $1\frac{1}{2}$ below H), P($4\frac{1}{2}$, in H, $1\frac{1}{4}$ before V), and Q($5\frac{1}{4}$, 1 before V, $1\frac{1}{2}$ above H).

Locate on the flat the following points:

5. A(in *III*, 1 to right, $\frac{3}{4}$ from V, 2 from H),
   B(in *II*, $2\frac{1}{2}$ to right, 1 from H, 1 from V),
   C(in *I*, 4 to right, $1\frac{1}{4}$ from V, $\frac{3}{4}$ from H).
6. D(in H, $1\frac{1}{2}$ behind V, $1\frac{1}{2}$ to right),
   E(in *III*, 3 to right, 1 from V, $\frac{1}{2}$ from H),
   F((in V, 1 above H, $4\frac{1}{2}$ to right).
7. G(in *I*, $1\frac{1}{2}$ to right, 1 from V, 1 from H).
   H(in *III*, 3 to right, 2 from V, $1\frac{1}{4}$ from H),
   K(between *II* and *III*, $4\frac{1}{2}$ to right, 1 from V).
8. M(in *IV*, $1\frac{1}{2}$ to right, $\frac{1}{2}$ from H, $\frac{1}{2}$ from *IV*),
   N(between *III* and *IV*, 3 to right, 2 from H),
   O(in *III*, $4\frac{1}{2}$ to right, 2 from V, 1 from H).

LINES

15. Until the subject of curved lines is studied, the word "line" will be taken to mean a **straight line.** Two points determine a line, and any line that passes through two points, contains those points. Further, the projections of that line will contain the projections of those points.

Theorem IV. Two projections of a line will, in general, determine the position of that line in space.

Proof.—Pass a plane through the V-projection of some line, and make it perpendicular to V. Also pass a plane through the H-projection of the same line, and make the plane perpendicular to H. These planes will intersect in a line, and that line must be the given line.

Theorem V. If a line is parallel to H, its H-projection is parallel and equal to the line, and its V-projection is parallel to GL.

Prove This.—What would be the statement if the line were parallel to V?

Theorem VI.– If a line is parallel to H and V, both of its projections will be parallel to GL.

Theorem VII.—If a line is perpendicular to H, its H-projection will be a point, and its V-projection will be perpendicular to GL.

What would be the statement for a line perpendicular to *V*?

Theorem VIII.—If a line is oblique to both H and V, both projections will be oblique to GL, except when the line lies in a profile plane.

16. Note.—An oblique line, lying in a profile plane, must be limited by two points; otherwise it is indeterminate. This is so, because both projections are perpendicular to *GL*, and therefore lie in the same projecting plane, hence cannot be determined by the intersection of two projecting planes (see Theorem III). The projections of any two points on the line determine the line.

PROJECTIONS ON THE PROFILE PLANE

17. When a line is so situated that its *H*- and *V*-projections are both perpendicular to *GL*, its true relation to *H* and *V*, as well as to other lines like itself, can only be shown by the **Profile.** This plane, called *P*, may be taken anywhere, either to right or

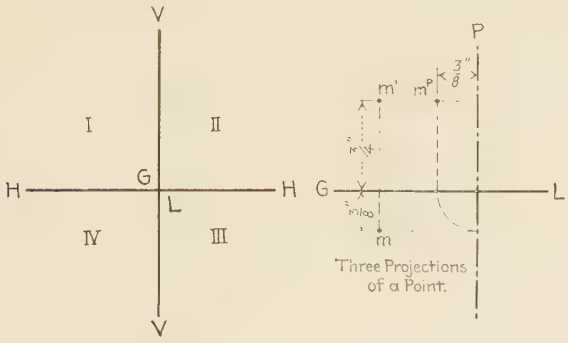


Fig. 8. Fig. 8a.

left, but is usually placed on the right of the problem, allowing sufficient space for its revolution into *V* in either direction. Its *V*-trace is taken for the axis.

When *P* is revolved into *V*, the effect, or view, is the same as when the observer looks at the plane in the direction of *GL*. Fig. 8 shows *H* and *V* in their perpendicular relation, viewed from the right side, with the four space angles in their relation to *H* and *V*. From this it is evident that in rotating the profile,

the V-trace of P becomes the projection of V itself, and that GL becomes the projection of H. The profile of any point is then shown in space its respective distances from H and V.

Fig. 8 (a) shows the three projections of a point M, $\frac{3}{4}$ in. above H, and $\frac{3}{8}$ in. in front of V. The P-projection is obtained by projecting m′ and m to x and y on the respective traces of P, and then rotating P about its V-trace, as in the figure. M shows in the P-projection that it is $\frac{3}{4}$ in. above H, and $\frac{3}{8}$ in. in front (to the left) of V.

If the point lies in any other angle, the revolution must be made in accordance with its location. The V-projection will always move in a line parallel to GL, whether it is above or below H. The profile will be that much above or below H. The H-projection will revolve the same as in Fig. 8a if the point is in front of V; that is, if the H-projection is below GL, in either angles I or IV. If it is in II or III, its H-projection is above GL, and the profile rotation will be clockwise, and be projected from GL to the projector from its V-projection.

Note.—The profile is seldom necessary to the solution of problems. It is chiefly used in problems where there are lines whose projections are perpendicular to GL, and planes, whose traces are parallel to GL.

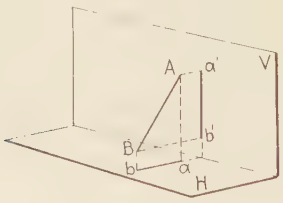


Fig. 9.

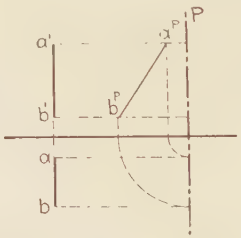


Fig. 9a.—Flat projection.

18. Fig. 9 shows a line, AB, in a profile plane, and its projections on H and V, in picture form. Fig. 9 (a) shows its flat representation, three projections being given. It is obvious that the profile is necessary for an adequate conception of the line.

Theorem IX.— **If a line lies in V, its V-projection is the line tself, and the H-projection is in GL.**

Why? What is the statement when the line lies in H?

Theorem X. If a point or line line lies in P, its V-projection will be in the V-trace of P, and its H-projection will be in the H-trace of P.

19. EXERCISES IN REPRESENTATION OF LINES

9. Draw the line A(5−1+3) B(2½+1−3), and tell where it travels.
10. Draw and designate the lines: A(⅞+1−¾) B(2+½−1½), C(2¼+2+1) D(3½+1+2), E(4−1−½) F(5½−½−2).
11. Draw and designate: G(1−1+1) K(2−½+½), M(2½, 0+1) N(3¾, 0+2), O(4½+1, 0) P(5½−1, 0).
12. What kind of lines are these? Draw and designate: A(1+1−½) B(2+1−1½), C(2+¾−1) D(3¼+¾+1), E(4+2−½) F(4+½−1½), G(4½+2−1) K(5½+1−1).
13. What kind of lines are these? Draw and designate: A(¾+1−1) B(¾+2−1), C(1½+¾−2) D(1½+¾−½), E(2+1½−1) F(3−1−1), G(4½+½−1) K(5½+½−½).
14. Tell what angles these lines traverse: P(½+2+1) O(1½−1−1½), N(2½+1−2) M(3½−1−1), L(4½−1+2) K(5½−½−1).
15. Tell what angles these lines traverse: A(½+1+1) B(1½−1−1), C(2½−½−1½) D(3½+1½+½), E(4½+1−1) F(5½−2+1).
16. From A(1+1−1) draw a line 2 in. long, parallel to V and inclined 45° to H. Does it touch H? Does it touch V?
17. From B(2+1½−1½) draw a line parallel to H and inclined 30° to V. How far on this line is it to V?
18. Draw the H-, V- and P-projections of the lines A(1+2−1) B(1+½−1½) and C(2½+1½+1½) D(2½+1+½). Locate P at 4½ in.
19. From K(1¾−2−1) draw a line parallel to V, and 45° to H, 4 in. long, upward toward the right. In what angle does it terminate?
20. From M(2−2−1) draw a line parallel to H, 30° to V, 3 in. long, toward V to the right. In what angle does it terminate?

REPRESENTATION OF PLANES

20. A plane other than H, V and P is shown by its **traces**; that is, its **intersections** with H, V and P. Usually the H and

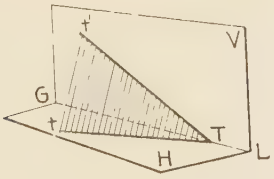


Fig. 10.

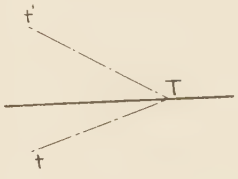


Fig. 10*a*.

V traces are sufficient, and the P-trace is seldom used. Fig. 10 shows an oblique plane, t′Tt (spoken of as the plane T), which cuts V in the line Tt′, its **V-trace,** and H in the line Tt, its

H-trace. Fig. 10*a* shows the flat representation of the same plane. Tt′ is one line of the plane T, and Tt is another line of the same plane, and the plane is completely determined by them.

Theorem XI.—The two traces of a plane completely determine the plane.

Note.—The student must on no account think that Tt and Tt′ are projections of the same line. Tt lies wholly in *H*, and its *V*-projection is in *GL;* hence, **if a point is assumed in Tt, its V-projection is in GL.** Tt′ lies wholly in *V*, and its *H*-projection is in *GL;* hence, **if a point is assumed in Tt′, its H-projection is in GL.**

21. Theorem XII.—The two traces of any plane must both be parallel to GL, or must both intersect GL, and in one point.

For *GL*, being common to *H* and *V*, must either intersect or be parallel to all lines in *H* and *V*. Tt is in *H*, and Tt′ is in *V*, therefore they must be parallel to *GL* or intersect it, and the intersections can only be in one point, because *GL* can only pierce T in one point. If the plane is parallel to *GL*, both traces must be parallel to *GL*. Why?

Note.—The traces do not end at *GL*, but are indefinite in length. The plane T occupies part of every space angle.

Theorem XIII.—If a plane is parallel to V, it has only one trace, the H-trace parallel to GL. What would be true of a plane parallel to *H?*

Theorem XIV. A profile plane is a plane perpendicular to N and V, and its traces are both perpendicular to GL.

Theorem XV. A plane perpendicular to V, but oblique to H, will have its **H-trace perpendicular to GL.** For example, see the lid of a box, as it takes different positions. What would be true of a plane perpendicular to *H?* See a swinging door.

Theorem XVI. If a plane passes through GL, it can only be satisfactorily shown by its P-trace. Both *H*- and *V*-traces are in *GL*, and therefore the plane must have either a profile trace or a point in space to determine it. All points in such a plane are projected on *P* in the *P*-trace of that plane. Why?

Theorem XVII.—Two parallel planes will have parallel traces on any plane of projection.

Give a reason for this from Solid Geometry. Is the converse of this always true? Take two intersecting planes parallel to *GL*.

Theorem XVIII.—Two planes perpendicular to each other do not in general have their traces perpendicular. The exceptions are, (1) a profile plane and a plane parallel to *GL;* and (2) two perpendicular planes that are also perpendicular to *H*, will have their *H*-traces perpendicular.

Note.—Planes perpendicular to *H* or *V* are called *projecting planes,* because they project every point and line in themselves on their traces on the plane to which they are perpendicular.

NOTATION OF PLANES

22. As the planes are represented by their traces, it is sufficient to give the notation of the traces to determine the plane.

Take the plane T in Fig. 10, a plane oblique to *GL*, and therefore to *H* and *V*. Its traces are t′T and Tt, three different points, t′, T and t, respectively in *V*, *GL* and *H*. It will be noticed that t′ and t are independent points, and would be designated (except for convenience) by different letters as points in space. The plane would be written in full t′(1 + 2, 0)T(5, 0, 0)t(1, 0 − 1). But, by omitting the zeros and the repetition of letters, we can reduce it to simpler terms, thus:

$$T(1+2)5(1-1)$$

Rule.—In the notation of a plane, the *V*-projection of a point in the *V*-trace is given first, the point where the traces intersect *GL* second and unbracketed, and the *H*-projection of a point on the *H*-trace third.

When a plane is parallel to GL it has no intersection with *GL;* that is, its intersection is at infinity, and it is then written $T(+2)\,\infty\,(-1)$, the infinity sign indicating that the traces are parallel to *GL*, and the *V* and *H*-traces 2 in. above and 1 in. below *GL* respectively. Both could be above or both below *GL*, or the signs could be opposite to those above.

Planes oblique to GL, but not intersecting within the problem boundaries, have a special notation, thus:

$$T(1+2)(5+1\tfrac{1}{2})(1-1\tfrac{1}{2})(5-1\tfrac{1}{8}).$$

It can be easily seen that this is giving the notation of two points on each trace.

Note.—For points and lines we employ the first letters of the alphabet, and for planes the latter letters.

EXERCISES IN REPRESENTATION OF PLANES

23. Draw the traces of the following planes, and state their relations to H, V and P.

Use a dot-and-dash line for all traces.

21. $T(1+2\frac{1}{2})3(3-2\frac{1}{2})$, $S(3\frac{1}{2}+3)3\frac{1}{2}(5\frac{1}{2}+1)$.
22. $Q(1+2\frac{1}{2})3(1+1)$, $R(3\frac{1}{2}+2)5\frac{1}{2}(3\frac{1}{2}-1)$.
23. $S(+1\frac{1}{2})\infty(-2)$, $T(5-1)3(1+1)$.
24. $U(1+2)(5+3)(1+1)(5+1\frac{1}{2})$, $X(-2)\infty(-1)$.
25. Draw the traces of the H and V projecting planes of the line $A(1+2-\frac{1}{2})$ $B(3+1+1\frac{1}{2})$, and the line $C(4-1\frac{1}{2}+1\frac{1}{2})$, $D(5\frac{1}{4}-\frac{1}{4}-\frac{1}{2})$.
26. Show the P-traces of the planes $Q(+2)\infty(+1)$, $R(-1\frac{1}{2})\infty(+1)$ $S(+1\frac{3}{4})\infty(-\frac{1}{2})$, and $T(-2\frac{1}{2})\infty(-1)$.
27. Find the P-trace of a plane T through GL and $A(1+1\frac{1}{2}-\frac{1}{2})$, and the P-trace of S through GL and $B(2\frac{1}{2}-1-\frac{1}{2})$.

Note.—To find the P-trace of any plane, take the plane P at any convenient place, where it will intersect the H and V-traces. Then treat the line connecting these intersections (the P-trace) as you would any other profile line, and revolve it as in Art. 17.

Theorems Relating to Lines and Planes

24. Theorem XIX.—**Two lines parallel in space have parallel projections on any plane.**

Theorem XX. Two lines perpendicular in space do not ordinarily have their projections perpendicular.

Special Case. If one of these lines is parallel to H, the H-projections of the two lines will be perpendicular.

Fig. 11 shows a line AB, parallel to H, and BC (any one of four different lines), whose H-projection is perpendicular to ab. Any one of the lines BC will be perpendicular to AB.

Proof.—If AB is parallel to H, any plane perpendicular to AB will be perpendicular to H, and will therefore be a projecting plane on H of all lines contained in it. Let the student finish the reasoning to show that all lines, whose H-projections are bc, will be perpendicular to AB.

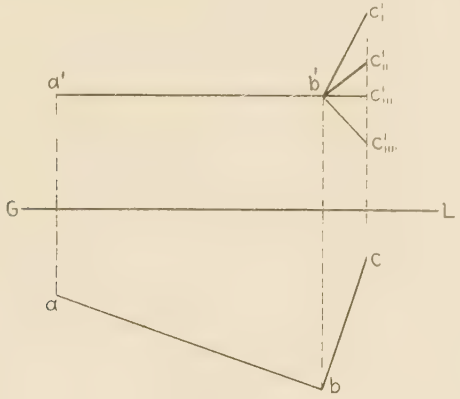


Fig. 11.

25. Theorem XXI.—Two intersecting lines in space must have their H and V-projections intersect in points in a line perpendicular to GL. Prove.

Theorem XXII. Two intersecting lines, both parallel to H, are projected on H in lines parallel and equal to themselves, and the included angle will be equal in projection to the angle in space. Prove.

Theorem XXIII.—A line, that is parallel to an oblique plane, is not, in general, projected parallel to the traces. The exception is only possible when the traces are both parallel or both perpendicular to GL.

Theorem XXIV. A line, that is perpendicular to a plane, always has its projections perpendicular to the respective traces of the plane.

In Fig. 12 let BC be perpendicular to T. To prove that bC is perpendicular to tt, the H-trace of T.

Let fall a perpendicular, Bb, from B to H. The lines BC and Bb determine a plane perpendicular to both H and T, and therefore to their intersection, tt. As the plane is perpendicular to tt, its intersection bC will be; that is, the H-projection is perpendicular to the H-trace. The same could be proved for the V-projection and V-trace.

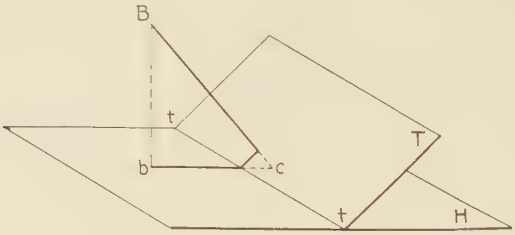


Fig. 12.

Theorem XXV. —Conversely, a line, whose projections are perpendicular to the respective traces of a plane, is perpendicular to the plane, with one exception.

Exception.—If the plane is parallel to GL, the line, whose projections are perpendicular to its traces, may not be perpendicular to the plane. Prove this by the P-trace of such a plane, and the P-projection of such a line, rotated.

26. Theorem XXVI.—A line in any plane pierces H in the H-trace of the plane, unless the line is parallel to H. Similarly, it will pierce V in the V-trace of the plane in which it lies. Prove.

Theorem XXVII. If a line is parallel to H, it is parallel to the H-trace of any plane in which it lies. Similarly for V. Prove.

Theorem XXVIII.—Any plane that is passed through one of two parallel lines, either contains or is parallel to the other. Let the student prove this with a figure.

EXERCISES IN REVIEW

27. Directions.—All exercises will be drawn in a space 6 in. × 7 in., with GL horizontally across the middle.

Letter all points carefully, using small letters for all projections, a′, b′, etc., for V-projections, a, b, etc., for H-projections, a^P, b^P, for P-projections.

Use the following conventions for all lines, etc.

Given lines—solid, medium,	$\frac{1}{64}''$
Required lines—solid, heavy	$\frac{1}{32}''$
Auxiliary lines—solid, light	very fine.
Invisible lines—dotted, medium	$\frac{1}{64}''$
Projectors—dotted, light	very fine.
Paths of Revolution—the same.	
Traces of Given planes—medium	$\frac{1}{64}''$
Traces of Required planes—heavy	$\frac{1}{32}''$
Traces of Auxiliary planes—light	very fine.

All planes are considered transparent, and there will be no invisible lines until problems involving solids are given. || and ⊥ are abbreviations for parallel and perpendicular.

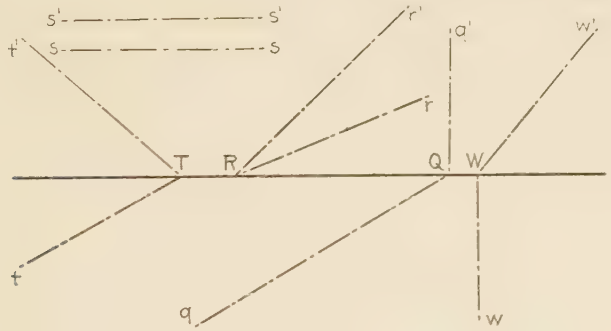


Fig. 13.

28. Through A(1+1½−1) draw a line AB || C(2+½+1) D(3½−1−½).
29. Through M(4½−1+2) draw a line MN || and equal to K(1+1¼−¾) O(2¼−½−2).
30. From D(5+2−1) draw two lines intersecting the line A(2−1+1½) B(3¼−½−½) in points other than A and B.
31. Draw two lines AB and CD intersecting in the point E(3−2−½). Connect these two lines with a third.
32. Draw four lines intersecting A(2−1+¾) B(3−2+¾) at right angles to it.
33. Through O(4+2−1) draw a line ⊥ to T(1+2)4(1−3).
34. Draw the traces of a plane S ⊥ to A(3+2−1¼) B(4+1−½).
35. Is M(3+1½−½) N(3+½−1⅛) ⊥ to T(+1) ∞ (−¾)? If not, draw the traces of a plane, S, that is ⊥ to MN.

36. Draw the P-projections of the following points: A($\frac{3}{4}+1+1$), B($1\frac{1}{2}-2+\frac{3}{4}$), C($2\frac{1}{4}+1\frac{1}{4}-\frac{5}{8}$), D($3-1-\frac{1}{2}$), and E($3\frac{1}{2}$, $0+\frac{1}{8}$). Place P at $4\frac{1}{2}$.
37. Find the P-projections of the following lines: M($\frac{3}{4}+1\frac{1}{4}+1$)N($\frac{3}{4}-\frac{1}{2}-\frac{3}{4}$) and O($2+1\frac{1}{4}-\frac{1}{2}$)P($2-1-\frac{1}{2}$). Profile at $4\frac{1}{2}''$.
38. Find the P-projections of the lines E($1-1-1$)F($1-\frac{1}{2}+1$) and G($2+2+\frac{3}{4}$)H($2+1-1$), P at $4\frac{1}{2}$.
39. Write the notation of the planes in Fig. 13, and describe them. Use your own dimensions.
40. Draw through O($2+1-\frac{1}{2}$) a line, OP, $2''$ long, || to H, and oblique to V, and a line ON, $2\frac{1}{2}''$ long, || to V, and oblique to H.
41. Let A(2, $0-1$)B($2\frac{1}{2}$, $0-2$) be the diagonal of a cube resting on H. Draw the projections of the cube, and the traces of the six planes bounding it.

CHAPTER II

PROBLEMS RELATING TO POINTS, LINES, AND PLANES

28. General Hints.—In the solution of the problems that follow, we (1) analyze the operations in space, or on paper, as the case may require, (2) offer proof of the correctness of this analysis, when it seems necessary, and (3) show the construction of a typical problem "on the flat." The analysis is the important part to learn, and should be thoroughly learned, not through memorizing the words or the figure, but by visualizing the condition in space, and imagining the process. The analysis furnishes the "machinery" for the work on the flat, and the worker should be adept with his tools. This can be done only by working exercises, of which there is an ample supply given. No attempt should be made by the student to see his particular exercise **in space,** but the work should be laid out on the flat, and worked according to the analysis. Then the space relations will be clear.

29. Problem 1.—To find the points in which a line pierces H and V.

Analysis.— (1) Extend the V-projection to GL. (2) From this point erect a perpendicular (projector) to the H-projection, or its extension. (3) The point of this intersection is the H-piercing point.

Interchanging H and V in the analysis will obtain the V-piercing point.

Proof.—Since all points in H have their V-projection in GL, this point must be in H, and therefore the point where the line pierces H.

Construction.—Let AB (Fig. 14) be any oblique line.

1. Extend a′b′ to c′ on GL.
2. Draw the projector to c on the extension of ab. C is the H-piercing point.
3. Extend ab to d on GL.
4. Draw the projector to d′ on the extension of a′b′. D is the V-piercing point.

Axioms.— 1. A line parallel to V has no V-piercing point, therefore its H-projection must be parallel to GL.

2. A line parallel to H has no H-piercing point, therefore its V-projection must be parallel to GL.

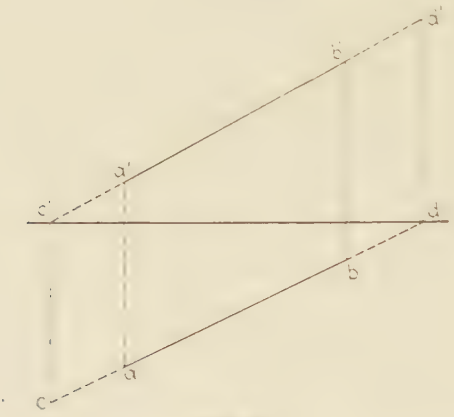


Fig. 14

30. Problem 2. -To find the H- and V-piercing points of a line in a profile plane.

Analysis.— 1. Project the line on P, located (preferably) on the right. See Fig. 8*a*.

2. Revolve P about its V-trace into V. (Determine the P-projections of two points of the line. See Fig. 8*a*.)

3. Extend the P-projection of the line to H and V, which are represented respectively by GL and the P-trace.

4. Project these points back to the original projection of the line, or its extension.

Proof.—In rotating P into V, all points in H will project on GL, and all points in V will project on the V-trace of P.

Construction.—Let AB (Fig. 15) be a line in a profile plane.

1. Project it on $P(a^Pb^P)$, and revolve it as shown.

2. Extend a^Pb^P to $H(GL)$, and $V(Pp')$, to the points c^P and d^P.

3. Revolve C^P down to Pp and project it back to ab (extended), at C, the H-piercing point.

4. Project d^P back to d' on a'b' (extended), giving the V-piercing point.

Fig. 15*a* shows the problem when A is in *II* and B in *III*.

Rules.—1. The *H*-projections that lie below *GL* (points in *I* and *IV*) revolve counter-clockwise and return clockwise.

2. The *H*-projections that lie above *GL* (points in *II* and *III*) revolve clockwise and return counter-clockwise.

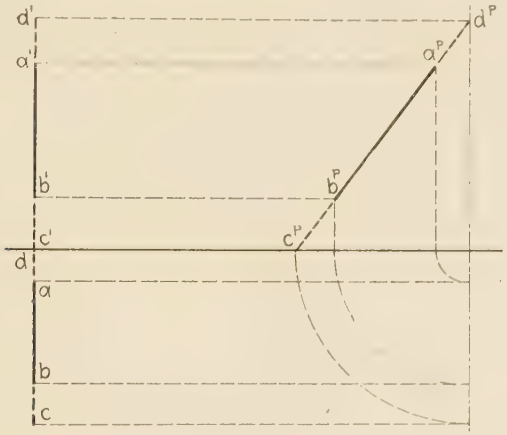


Fig. 15.

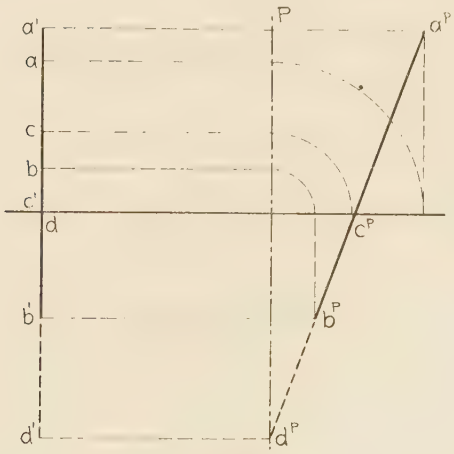


Fig. 15*a*.

3. All *V*-projections move parallel to *GL*.

Axiom.—Lines in profile planes, when projected on *P*, revolved into *V*, are shown in their true length, and in their true angles to *H* and *V*.

EXERCISES

31. Find the points in which H and V are pierced by the following lines, or their extensions. Tell what angles they traverse.

42. A$(1+1\frac{3}{4}-\frac{1}{4})B(2\frac{3}{8}+\frac{1}{2}-1\frac{1}{4})$, C$(4+\frac{1}{2}-\frac{1}{2})D(5\frac{1}{2}+1\frac{1}{2}-1\frac{1}{2})$.
43. E$(1+\frac{1}{2}+2)$F$(2\frac{1}{2}+1\frac{1}{2}-\frac{1}{2})$, G$(3-\frac{1}{2}-1)H(5-1-1\frac{3}{4})$.
44. M$(\frac{3}{4}+\frac{1}{2}+\frac{1}{2})N(2+1\frac{1}{2}+1\frac{1}{2})$, O$(4+1-\frac{1}{2})P(5+1-2)$.
45. K$(1+\frac{1}{2}-1)$L$(2+2-1)$, A$(4+1-1)$B$(5\frac{1}{2}-\frac{1}{2}+2)$.
46. E$(1,\ 0+1\frac{1}{2})$F$(2\frac{1}{2},\ 0-\frac{1}{2})$, G$(3-1\frac{1}{2}-\frac{1}{2})H(5+\frac{1}{2}+1\frac{1}{2})$.

Note.—For the following exercises one P-plane at $4\frac{1}{2}''$ from the margin will suffice for each exercise.

47. A$(1+2-\frac{1}{4})$B$(1+\frac{1}{2}-1)$, C$(2+1+\frac{1}{2})$D$(2-\frac{3}{4}+\frac{1}{4})$.
48. E$(1+\frac{1}{2}+\frac{1}{2})F(1+1\frac{1}{2}+\frac{1}{2})$, G$(2+\frac{3}{4}+\frac{1}{2})M(2-\frac{1}{2}-1\frac{1}{2})$.
49. K$(1+1-1)$L$(1-\frac{1}{2}-1)$, M$(2-\frac{1}{4}+\frac{3}{4})N(2-2+\frac{1}{4})$.
50. O$(1-\frac{3}{4}-1)$P$(1+1-\frac{1}{4})$, A$(2-1\frac{1}{2}+\frac{1}{2})B(2-\frac{1}{2}-\frac{1}{2})$.
51. C$(1+1\frac{1}{2}+\frac{1}{2})D(1+\frac{1}{2}+1)$, E$(2-\frac{1}{2}-1)F(2-2-\frac{1}{4})$.

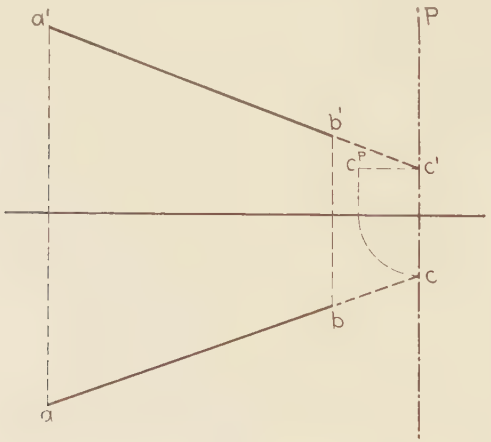


Fig. 16.

32. Problem 3.—To find the piercing point of a line with a profile plane, and show it in P-projection.

Analysis.—1. The point in which the line pierces P will have its H- and V-projections where the P-traces cut the H- and V-projections of the line.

2. Revolve this point as in Problem 2.

Proof.—P is a projecting plane on H and V, hence any point in the plane will be projected on its traces. (Theorem X, Art. 18.)

Construction.—Let it be required to find the P-piercing point of AB (Fig. 16).

1. Extend a′b′ to the P-trace at c′.

2. Extend ab to the P-trace at c. c is the required point.
3. Revolve P into V, locating c^P.

33. Problem 4.—To assume a line in a plane.

Analysis.—1. Assume a point in the V-trace.
2. Assume a point in the H-trace.
3. Connect the respective projections of the points.
4. The line will be contained in the plane.

Proof.—If any two points lie in a plane, the line joining them will lie wholly in the plane.

Construction.—Let t′Tt (Fig. 17) be any plane.
1. Assume a′ on t′T, a will be in *GL*. (Theorem IX, Art. 18.)
2. Assume b on tT, b′ will be in *GL*.
3. Draw a′b′ and ab.
4. AB will be contained in T.

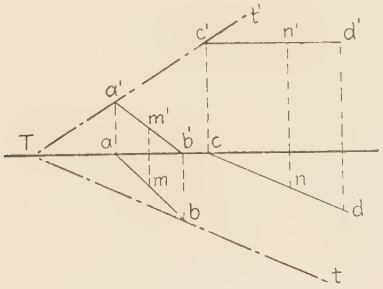


Fig. 17.

34. Second Case.—If it be desired to draw a line in T parallel to H, assume a point c′ on t′T, and draw c′d′ parallel to *GL*. Project c′ to c on *GL*, and draw cd parallel to Tt. (Theorem V, Art. 15, and Theorem XXVII, Art. 26.)

35. Problem 5.—One projection of a point in an oblique plane being given, to find the other projection.

Analysis.—1. Through the given projection draw a line, assuming it to be one projection of a line in the given plane.
2. Locate the other projection of the line. (Problem 4.)
3. Project the given point to the other projection of the assumed line, and this will be the required point.

Proof.—If a line lies in a plane, all its points are in the plane.

Construction.—See Fig. 17.
1. Assume m′ or n′.
2. Through m′ draw a′b′, or through n′ draw c′d′.

3. Locate ab or cd (Problem 4).

4. Project m′ to m on ab, or n′ to n on cd. M and N will lie in T.

36. Hint.—It is usually more satisfactory and speedy to use a parallel line, like CD, rather than an oblique line, like AB. Sometimes, as in the case of a plane parallel to *GL*, the oblique line only can be used conveniently.

Note.—If a point is to be assumed in a projecting plane, the point may be indeterminate; that is, if the assumed projection is taken on the projecting trace, there may be an infinite number of points having that projection, and lying in the plane.

37. Problem 6.—To find the traces of a plane determined by three given points.

Analysis.—1. Join the points, two and two, by lines.

2. Find the piercing points of the lines.

3. Draw the *H*-trace through the *H*-piercing points, and the *V*-trace through the *V*-piercing points.

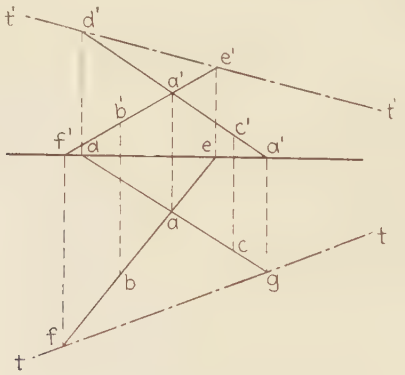


Fig. 18.

Proof.—The lines have their piercing points in the respective traces of the plane, therefore the lines (and all their points) lie in the plane thus determined.

Construction.—Fig. 18 shows three points, A, B and C.

1. Join the respective projections of AB and BC.

2. Find the piercing points, D, E, F and G.

3. Through d′ and e′ draw t′t′, and through f and g draw tt. These will be the required traces.

Check.—1. The line BC should have its piercing points in the respective traces of T, and

2. the traces should meet in a point on GL, or be parallel to GL.

Hints.—1. Sometimes the points may be so placed that the piercing points are not all to be found within the limits of the problem. In such a case, an auxiliary line may be drawn from any of the three points to any point on the line joining the other two. For instance, if such a proceeding were necessary, a line could be drawn from B to any point, M, on the line AC. BM would lie in the required plane, and its piercing points would help to determine the traces.

2. Ordinarily, only three piercing points are necessary to be found, because when the H-trace is found, its intersection with GL may be joined to one of the V-piercing points.

38. If two intersecting lines are given.—This makes no change in the problem.

39. If two parallel lines are given.—Find their piercing points and draw the traces, as in the previous case.

40. If a point and a line are given.—The point may be connected with any point in the line, thus making two intersecting lines, or through the point a line may be drawn parallel to the given line, thus making two parallel lines. (Theorem XIX, Art. 24.)

EXERCISES

41. Draw the H- , V- and P-projections of the following lines. Place P at $4\frac{1}{2}''$ in each exercise.

51. A$(2+\frac{1}{2}-1)$B$(3\frac{1}{2}+1\frac{1}{2}-1)$.
52. C$(1\frac{1}{2}+1+1\frac{1}{2})D(3\frac{1}{2}+1+1\frac{1}{2})$.
53. E$(2+1\frac{1}{4}-1)$F$(3\frac{1}{2}+\frac{1}{2}-\frac{1}{4})$.
54. K$(2+\frac{1}{2}+2)$L$(3\frac{1}{2}-1+\frac{1}{2})$.

Locate the following lines in the planes given:

55. AB, any oblique line in $\top(4\frac{1}{2}+2)2(4\frac{1}{2}-1\frac{1}{4})$.
56. C$(2+2,$ x$)$D$(3+1,$ y$)$in T (Ex. 55). State values of x and y.
57. E$(2-1,$ x$)$F$(1-\frac{1}{2},$ y$)$ in T (Ex. 55). State values of x and y.
58. GH, a line || to H, $1\frac{1}{4}''$ above H in T (Ex. 55).
59. KL, a line || to V, $1''$ behind V in T (Ex. 55).
60. MN, a line in III, || to t′T (Ex. 55).
61. O$(4+1,$ x$)$P$(2,\ 0,\ 0)$ in T (Ex. 55). State value of x.
62. Find the locus of all points in T (Ex. 55), $1''$ above H, also of those $\frac{1}{2}''$ below H.
63. Find the locus af all points in T (Ex. 55), $1''$ behind V, also of those $1''$ in front of V.

64. Locate the following lines in S(+2)∞(−1½): A(1+1½, x)B(2+½, y), C(2½, z+½)D(3½+1, q), and EF, the locus of all points ½″ in front of *V*. (Hint. It is not necessary to use a profile plane to obtain any of these lines, though it may be found useful for the line EF.)
65. Locate the following lines in R(+1)∞(+2): G(1+½, x)H(2, y−½), K(3, z+1)L(4+1½, q) and MN, locus of all points ½″ below *H*.

Locate the following points in the planes given:

66. A(4+1, x), B(3+2, y), C(2, z−2), and D(1−2, q) in the plane T(5+3)1½(5−2).
67. E(2, x+½), F(3½, y−1), G(4, z+½), and H(5, q−½) in the plane R(1+2)3½(6−2).
68. M(1, x+½), N(2−1, y), and O(4, z−1) in the plane S(+1½)∞(−1).

Find the traces of the planes determined by the following points and lines:

69. A(3+1−½), B(3½+½+1¼), and C(4−½+1).
70. D(2—2+¾), E(3¼+½−1½), and F(4½+1−½).
71. K(2+1½−¾), L(3+¾−¾), and M(5−1−½).
72. N(2+1−1), O(3−1+½), and P(4+2−½).
73. A(1½+3+1)B(2½+1−1), and BC(3−2−½).
74. The triangle D(2+1−1½)E(2½+1½−½)F(3½+½−½).
75. The parallelogram K(1½+1−⅝) L(2+1⅝−⅜) O(2⅝+⅜−1⅝) N(3⅛+1−1⅜).
76. With the line A(2, 0−1)B(2½, 0−2) as a base diagonal, erect a cube standing on *H*. Find the traces of all the planes bounding the cube.
77. With the line C(3, 0−2)D(3½, 0−1) as a base diagonal, erect a square pyramid of 2″ altitude. Draw its projections and the traces of the four planes of the slanting facse.

42. Problem 7.—To pass a plane through a given point parallel to two given lines.

Analysis.—1. Draw through the given point two lines, parallel respectively to the two given lines. (Theorem XIX, Art. 24.)

2. Pass a plane through the lines thus drawn.

Proof.—Referring to Theorem XXVIII, Art. 26, we see that the plane thus found must be parallel to both of the given lines, because it contains lines parallel to each of them.

Construction.—Let *A*, Fig. 19, be the given point, and BC and DE be the given lines.

1. Through a′ draw m′n′ parallel to b′c′, and o′p′ parallel to d′e′. Through a draw mn parallel to bc, and op parallel to de.

2. Find the *H*- and *V*-piercing points of MN and OP.

3. Through these piercing points draw the traces of T.

Note.—If the two given lines are parallels, an infinite number of planes may be passed through the point parallel to the two lines, and the problem is therefore not definite.

43. Problem 8.—Through a given line to pass a plane parallel to another given line.

Analysis.—1. Through any point in the given line draw a line parallel to the second given line.

2. Pass the required plane through the two intersecting lines thus drawn.

Let the student supply the proof and make the construction.

Note.—If a plane be passed through the second line parallel to the first, this plane will be parallel to the first plane.

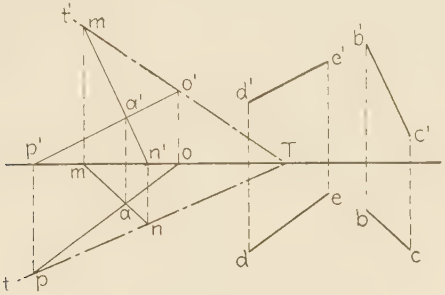


FIG. 19.

EXERCISES

4. Pass planes through the following points parallel to the lines given.

78. Through O(3+1−1) || to A(1+1½−1)B(2+½−2) and C(4+1−1) D(5+1½−2).

79. Through M(4−½+1) || to E(1½+1¼−1¼)F(2½+½, 0), and G(3+½+1) K(4+1½+2).

80. Through A(3¼+2−1)B(4+1−½) pass the plane T || to C(1+½−½) D(2+1½−¼).

81. Through E(3+1½−½)F(4+1−¾) pass a plane S || to G(½+½−1) H(1½+1½−½).

82. Through K(3½+1−½)L(5+2−1) pass a plane || to M(3−1+1) N(4½−2+3).

45. Problem 9.—To revolve a point in space into H about a line contained in H.

Definition of Revolution.—To revolve a point about a line (called its **axis**) means to move the point in a plane perpendicular to the axis, keeping it at a constant distance from the axis. The path of the point is a circle, and the center of the circle is the point where the axis pierces the plane of revolution.

Process.—Examine Fig. 20, which is a picture of the two planes of projection, *H* containing the line BC, and the point

A being located in space. The projections of A are shown on V and H. The circle, in which A is to revolve, lies in a plane perpendicular to BC (by hypothesis, see foregoing definition), and is therefore perpendicular to H. (Prove by Solid Geometry.) This plane is therefore a projecting plane, and contains not only A, but its H-projection, a; and its H-trace runs through the projection, a, perpendicular to BC, and crosses BC at x, the center of the circle of revolution. Ax is then the **radius of revolution.** Ax is the hypothenuse of the right triangle Axa. A will revolve into H at the point a_1, which is upon the line ax a distance equal to the hypothenuse of Axa. The side Aa of Axais equal to a′y, by Theorem I, Art. 9.

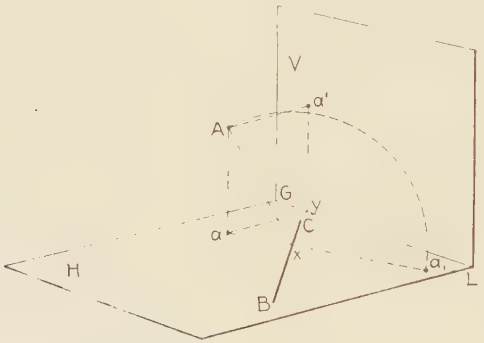


Fig. 20.—The revolution of a point.

Analysis.—1. Draw the trace of the plane of revolution through the H-projection of the point, and perpendicular to the axis.

2. Construct the right triangle, as follows: First, its base is the distance of the H-projection of the point to the axis, and second, its altitude the distance of its V-projection from GL.

3. Lay off the length of the hypothenuse of this triangle along the trace of the plane of revolution, measuring **from the axis** in either direction.

4. This point will be the revolved position of the point.

Proof.—The process, already given, contains the proof.

Construction.—Refer to Fig. 21.

Let A be the point, and BC the given line in H.

1. Through a draw a line perpendicular to bc.

2. Construct a right triangle axz using ax and a′y as sides.

3. Lay off the hypothenuse, xz, along xa (extended in either direction) **from the point** x.

4. a_1 will be the required position.

Notes.—1. To revolve a point into V about a line in V, make an interchange of H and V in the foregoing.

2. If the point is directly over the axis, the base of the triangle becomes zero, and the altitude and hypothenuse will be equal.

3. The "trace of the plane of revolution" and "the projection of the path of revolution" will be abbreviated hereafter to "the path of revolution."

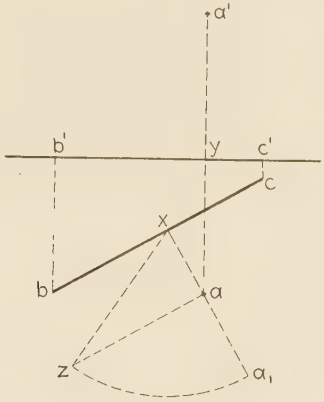


FIG. 21.

46. EXERCISES

82. Revolve A($2\frac{1}{2}+2-2\frac{1}{2}$) and B($5-1+\frac{1}{2}$) into H about C(1, 0−2) D(5, 0+2).

83. Revolve E($3+1\frac{3}{4}+1$) and F($4-\frac{1}{2}-1\frac{1}{2}$) into V about G(1−2, 0) H(5+2, 0).

84. (a) Revolve K($1+2-\frac{1}{2}$), L($2+\frac{1}{2}+1\frac{1}{2}$), M(3−1−1), and N($4-\frac{1}{2}+2$) into H about GL.
(b) Revolve them into V about GL.

85. (a) Revolve M(2+1−1) into V about O(1+1, 0)N(3+2, 0).
(b) Revolve A(4+2+) into H about B($3\frac{1}{2}$, 0−2)C(5, 0−1).

86. Locate D(3+1, x) and E(4, y$-\frac{3}{4}$) in T(5+3)1(5−2). Revolve D and E(1) about t'T into V, (2) about Tt into H.

87. Locate F(2+2, x) and K($3\frac{1}{2}$, y$-1\frac{3}{4}$) in S($+1\frac{1}{2}$)∞(−1). Revolve F and K about the H-trace of S into H.

88. Locate M(2+1, x) and N(4, y$-\frac{1}{2}$) in S($+1\frac{1}{2}$)∞($+\frac{3}{4}$), and revolve them into V about the V-trace of S.

89. Revolve two points, O and P, both in the V-trace of T (Ex. 86), O being $\frac{1}{2}''$ and P being $1\frac{1}{2}''$ above H, about the H-trace of T into H.

90. Revolve A($2+1\frac{1}{2}$, 0) about the H-trace of S($+1\frac{1}{2}$)∞($-\frac{3}{4}$) into H. Revolve B(4, 0$-\frac{3}{4}$) about the V-trace into V.

91. Revolve the triangle D(2+1−½)E(3−½−2)F(3½+½−1) about the H-trace of its plane into H. Will this be its true size?

92. Revolve the parallelogram K(2+1−1½)L(3+1½−1)O(4+1, 0) N(3+½−½) about its V-trace into V. Will this give its true size?

47. Problem 10.—To revolve a point through any angle about a line perpendicular to H or V.

Analysis.—(About a line perpendicular to H.)

1. The point moves in a circle parallel to H, which is projected on H in a circle of the same size.

2. Draw the circle, using as radius the distance from the H-projection of the point to the H-projection of the line.

3. Draw the V-projection of the circle. This will be a line through the V-projection of the given point, and parallel to GL. Why?

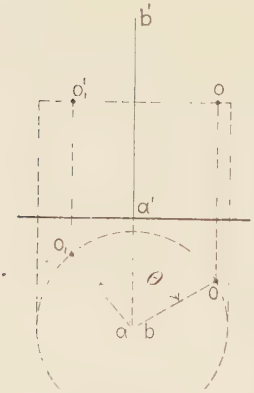


Fig. 22.

4. Lay off the required arc on the circle from the original position of the point (H-projection), locating the H-projection of the revolved position of the point.

5. Project this point to the V-projection of the circle.

Proof.—The proof of this depends upon the truth, that the projections of the circle are as given in the analysis. Make a drawing and give this proof.

Construction.—Let O (Fig. 22) be the given point, AB the given axis, perpendicular to H, and θ the given angle.

1. Draw the H-projection of the circle with a as its center and ao its radius.

2. Draw the V-projection of the circle, through o′, parallel to GL.

3. Move o through the given angle, θ, to o_1.

4. Project o_1 to o'_1, giving both projections of the point in the required revolved position.

48. Problem 11. To revolve a line through any given angle about a line perpendicular to H or V.

Analysis.—1. As the line is the sum of all points in it, take any two points of it and revolve them about the axis, as in Problem 10.

2. Join these revolved points, and the resulting projections will be projections of the line.

49. EXERCISES

93. Revolve A($1\frac{1}{2}+1-2$) 90° counter-clockwise about M(1, 0−1) N(1+2−1), and B($3\frac{1}{2}+2-1\frac{1}{2}$) 90° clockwise about P (4+1, 0) O(4+1−2). Also revolve B until it comes to H.
94. Revolve C($2+1\frac{1}{2}+1$) 45° clockwise about K(1+1−2) L(3+1−2), and M($4\frac{1}{2}+\frac{1}{2}-2$) N($5\frac{1}{2}+1\frac{1}{2}-1$) 135° counter-clockwise about D(4, 0−1) E(4+2−1).
95. Revolve A($2+\frac{1}{4}-1$) B($4+1\frac{1}{2}-2$) about P(3+1−0) O(3+1−3) into a position parallel to H.
96. Revolve E(2+2−1)F($3+\frac{1}{2}-2$) 75° clockwise about K($1+1-1\frac{1}{2}$) L($4+1-1\frac{1}{2}$).

50. Problem 12.—To find the true length of a line in space.

Discussion.—A line oblique to a plane is not projected **in its true length** on that plane. To be projected in its true length, a line must be parallel to that plane, or be contained in it. See Theorems V–X.

It follows, then, that any oblique line, in order to be measured, must be moved, without changing its actual length, into a position **parallel to or in** H or V.

Case I.—By revolving parallel to V.

Analysis.—1. Through any point in the given line (preferably one of its extremities) draw an axis perpendicular to H.

2. Revolve the line about this axis (Problem 11) until the H-projection is parallel to GL.

3. The line will then be parallel to V, and its V-projection will be the true length of the line.

Proof.—The line remains unchanged in revolving about its axis, because every point revolves about the axis in a series of parallel circles through the same angle. This means that every point remains in the same relation to the axis throughout the revolution, and therefore all points in the line remain in the same relation to each other, and the line is unchanged.

(**Note.**—The length of the H-projection is not changed, but that of the V-projection is changed.)

Construction.—Let AB (Fig. 23) be the given line.

1. Let CA be the axis, perpendicular to H.

2. Revolve b in a circle about a as a center, until ab_1 is parallel to GL.

3. Move b′ parallel to GL, until it intersects the projector from b_1.

4. Draw $a'b'_1$, which is the true length of AB.

Check.—Revolve AB about an axis perpendicular to V, until it is parallel to H, and this result should be equal to the first result.

51.—Case II.—By revolving into H.

Analysis.—1. Let the axis be the H-projection of the given line.

2. Revolve the line about this axis into the plane of the axis, H. (Problem 9, Note 2.)

The line will be in H, and therefore projected in its true length on H.

Proof.—By using the H-projection of the line for the axis, we make sure that the line, revolving about the axis, can be brought into H, in which the axis lies.

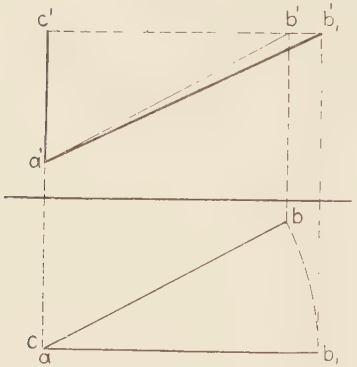


Fig. 23.

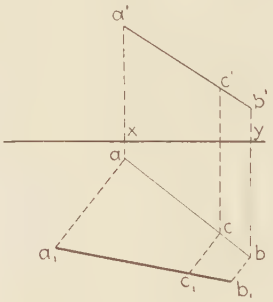


Fig. 24.

The analysis might be stated thus: **Fold** the projecting plane of the given line into H, about its line of intersection with H.

Construction.—Let AB (Fig. 24) be the given line, and ab the axis.

1. Lay off perpendiculars to ab from a and b. (See Problem 9, Note 2.)

2. Measure off on these perpendiculars a′x and b′y, giving the points a_1 and b_1.

3. Connect a_1 and b_1. This will be the required length.

Notes.—1. The line pierces H in its H-projection, and therefore the H-piercing point of the line lies in the axis, and will remain stationary in revolution. This fact may be used as a check on the work.

2. If the ends of the line are on opposite sides of the plane into which

they are revolved, the perpendiculars will have to be laid off on opposite sides of the axis.

Referring to Fig. 24*a*, we have AB, a line running through V from point in I to a point in II. The V-piercing point is at o between A and B. Taking a portion, BC, of AB, lying entirely on one side of V, and revolving it to $b'_1c'_1$, we find that b'_1, c'_1, and o' lie in one line, and continuing that line, we find that it meets A in revolved position at a'_1, on the opposite side of the axis.

3. In both of the foregoing cases of Problem 12 the planes V and H are interchangeable.

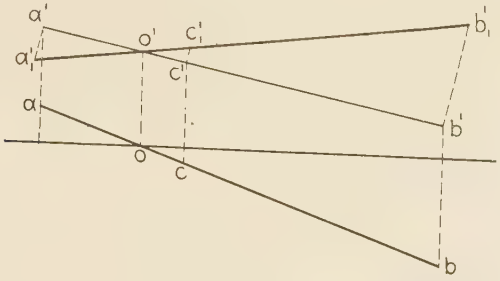


Fig. 24*a*.

52. Case III.—When the line is in a profile plane.

Analysis.—1. Project the line on P and revolve into V, as described in Problem 2.

2. The P-projection, thus revolved, is the true length of the line.

Proof.—The line is parallel to P, and is therefore projected on P in its true length.

Construction.—Exactly the same as employed in Problem 2, Figs. 15 and 15*a*.

Note.—Points in II and III will be rotated to the right side of P.

53. Problem 13.—To lay off a given distance on a given oblique line.

Analysis.—1. Revolve any definite portion of the line to show its true length, as in Problem 12.

2. Lay off the given distance.

3. Counter revolve the limiting points to the original position of the line.

Construction.—Use Fig. 24. Let it be required to lay off an inch from A on AB.

1. Revolve AB to a_1b_1.

2 Let a_1c_1 be 1 in. Lay it off from a_1.

3. c_1 being the other extremity of the line, must now be revolved back with the line into the original position. As all points of the line revolve in parallel circles, the return revolution will be made exactly as the outward revolution. The paths of revolution are perpendicular to ab, therefore C will move back to c'c, the same as A and B moved into H, in a line whose projection is perpendicular to the axis.

4. AC is then a part of AB 1 in. long.

Note.—Any method of measuring a line may be used in laying off this distance.

Axiom.—If a line be divided into any number of segments, the projections of these segments will be proportional to the segments in space. Thus, to bisect a line, **bisect its projections,** etc. Prove this by Plane Geometry.

54. EXERCISES

97. Measure A($1\frac{1}{2}+\frac{1}{2}-\frac{1}{2}$) B($4+2-1\frac{1}{2}$), (1) by revolving || to V; (2) by revolving into H.
98. Measure C($2+\frac{1}{2}-2$) D($4+2-\frac{1}{4}$), (1) by revolving || to H; (2) by revolving into V.
99. Measure E($2+\frac{1}{2}+\frac{1}{2}$) F($4+2+2$), (1) by revolving || to V; (2) by revolving into H.
100. Measure G($1+2+1$) H($4-1-1\frac{1}{2}$), (1) by revolving || to H; (2) by revolving into V.
101. Measure K($1+1\frac{1}{2}-1$) L($4-1+2\frac{1}{2}$), (1) by revolving || to V; (2) by revolving into H.
102. Measure M($2-\frac{1}{2}-\frac{1}{2}$) N($4-2\frac{1}{2}-2\frac{1}{2}$), (1) by revolving || to H; (2) by revolving into V.

Find the true length of the following lines. Place the Profile plane at $4\frac{1}{2}''$.

103. A($1+2-\frac{1}{4}$) B($1+\frac{1}{2}-1$), and C($2+1+1\frac{1}{2}$) D($2-1+\frac{1}{2}$).
104. E($1+\frac{3}{4}+\frac{1}{2}$) F($1+1\frac{1}{2}-\frac{1}{2}$), and G($2+1+\frac{1}{2}$) H($2-\frac{1}{2}-2$).
105. K($1+1-1$) L($1-\frac{1}{2}+1$), and M($2-\frac{1}{2}+\frac{3}{4}$) N($2-2+\frac{1}{4}$).
106. O($1-\frac{1}{2}-1$) P($1+2-\frac{1}{2}$), and A($2-1\frac{1}{2}+\frac{1}{4}$) B($2-\frac{1}{2}-1\frac{1}{2}$).
107. C($1+1\frac{1}{2}+\frac{1}{4}$) D($1+\frac{3}{8}+1$), and E($2-\frac{1}{2}-1$) F($2-2-\frac{1}{2}$).
108. Lay off on the line A($2+2-1\frac{1}{2}$) B($4+\frac{1}{2}-\frac{1}{2}$) the point C, $1\frac{1}{2}''$ distant from A.
109. Lay off on the line E($1+\frac{1}{4}+1\frac{3}{4}$) F($4+1\frac{1}{2}-1$) a point D, $2''$ from E.
110. Lay off on the line K($2+2-\frac{1}{4}$)L($2+\frac{1}{2}-1$) a point N, $1\frac{1}{2}''$ from L.
111. Lay off on the line A($2-\frac{1}{4}+\frac{5}{8}$)B($2-1\frac{3}{4}+\frac{1}{4}$) a point C, $3''$ from B.
112. Through D($1+1-1$) draw a line $2\frac{1}{2}''$ long, parallel to E($3-\frac{1}{2}+1$) F($4\frac{1}{2}+1-1\frac{1}{4}$).
113. M($2,\ 0-1$) N($3\frac{1}{2},\ 0-2\frac{1}{4}$) is the diagonal of the base of a right square pyramid, $2\frac{1}{2}''$ high. (a) Find the length of one of its slanting edges. (b) Find the length of a line running from a point $1\frac{3}{4}''$ high on one edge to a point $\frac{1}{2}''$ high on an adjacent edge.

114. A cable runs through a factory building from A($1\frac{1}{2}+2\frac{1}{2}-\frac{1}{2}$) to B($4\frac{1}{2}-\frac{1}{2}-2$) in the floor below.
(a) What is the length of the cable? Scale $\frac{1}{4}''=1$ ft.
(b) At what point in the floor (H) must the hole be drilled for it?
(c) A support from the wall (V) is required 6 ft. from the floor hole, measured on the cable. How far from the wall and the floor is the point of support?

115. Scale 1 in. = 100 ft. A rope conveyor is to be constructed from a point (M50ft.,0−160ft.) to a loading point on a hill 175 ft. high, and 225 ft. north, 55° east of the dumping point. How much rope must be purchased for a duplex belt, allowing 10 ft. for slack, pulleys, etc.?

116. Fig. 25 shows the plan and elevation of a plain hip roof. Draw it to any convenient scale.

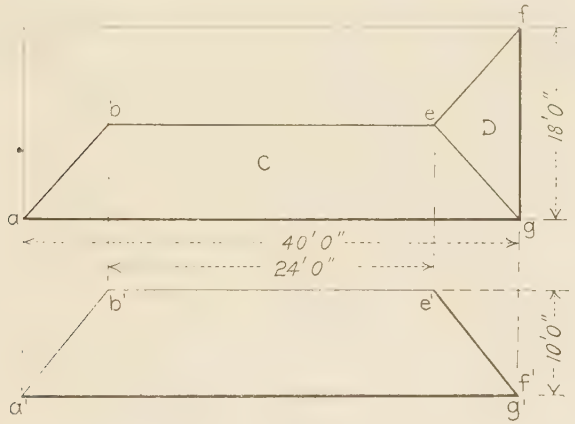


FIG. 25.

(a) Which is its H- and which its V-projection? What angle is it drawn in? Draw its P-projection.
(b) Locate the traces of the roof planes, C and D.
(c) What is the length of the hip rafter AB?
(d) Locate a point in the roof C, 3 ft. below the ridge, and 10 ft. to the right of the center.
(e) Locate a point on the rafter EF, 3 ft. from F.

117. Scale 1 in. = 20 ft. A 60-ft. stack, 3 ft. in diameter, rises from A(60 ft., 0−35 ft.), on the top of a flat roof (H). Five guy-wires are attached to the stack at 40 ft. above the roof, and are anchored at points on the roof, respectively, B(30 ft., 0−10 ft.), C(85 ft., 0, −7 ft. 6 in.), D(90 ft., 0−47 ft. 6 in.), E(70 ft., 0−67 ft. 6 in.), and F(25 ft., 0−52 ft. 6 in.). Draw the stack and guys, and measure the true length of the guys.

55. Problem 14.—To measure the angle between any two intersecting lines.

Two intersecting lines lie in one plane, and, if that plane be folded so as to coincide with H or V, the revolved projections will show the true relation of the lines.

Analysis.—1. Pass a plane through the two lines.

Note.—One trace only need be obtained.

2. Revolve the plane about its H-trace into H, or about its V-trace into V.

Note.—Revolving the various points in a plane about one of its traces is equivalent to revolving the plane.

3. The points are now shown in their true relation to each other, and the angle is in its true size.

Construction.—Let it be required to measure the angle between the lines AB (Fig. 26) and CD, which intersect at O.

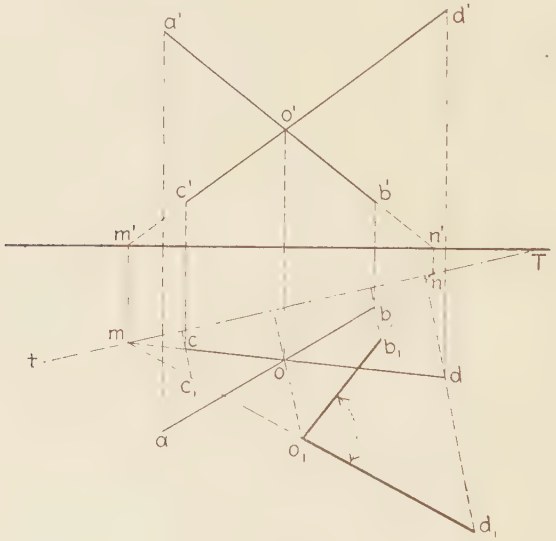


Fig. 26.

1. Find the piercing points, m and n, of the lines, and draw the H-trace, tt, through them.

2. Revolve O (by Problem 9) about tt into H to the point o_1.

3. Connect m and n with o_1, and the revolved positions of AB and CD are found.

4. The required angle is then either no_1d_1 or no_1m.

Proof.—In this construction, operation (3) may be explained, as follows: M and N are points in CD and AB respectively, and

O is in both. Hence if N, O, and M are revolved, both lines are revolved, because two points in any line determine that line fully. M and N, being in the axis, will not move in the revolution (see Art. 51, Note 1), and so, if they are connected with the revolved position of O, we shall have both lines revolved. The revolved position of any point in either line may be found by running a perpendicular to the trace through the *H*-projection of the point, until it intersects the revolved position of the line. The points b_1 and d_1 in Fig. 26 are located in this manner.

Notes.—1. Either angle between the lines may be considered the angle to be measured, but it is customary to choose the acute angle, unless otherwise specified.

2. It will sometimes happen that the piercing points of the lines, or some of them, will be inaccessible. Under such conditions, one or more auxiliary lines may be drawn through points in the given lines. These auxiliaries will lie in the plane, and may be so drawn as to determine the desired trace.

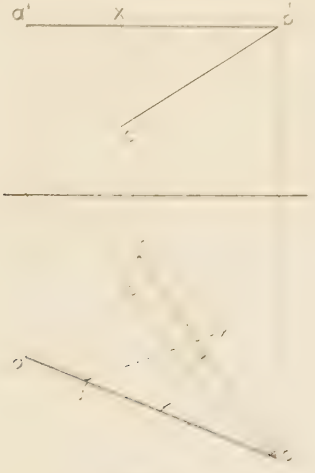


FIG. 27.

56. Problem 15. To measure an angle when one side is parallel to H.

Analysis.—1. Revolve the oblique line about the parallel line as an axis, until it is also parallel to *H*. This is a modification of Problem 11.

2. Both lines being parallel to *H*, the angle is projected on *H* in its true size. (Theorem XXII, Art. 24.)

Construction.—ABC (Fig. 27) is the required angle, with AB parallel to *H*.

1. Revolve C about AB as an axis into the same horizontal plane in which AB lies. To do this, run the path of revolution through c, and construct the right triangle for the radius, by using $c'x'$ and cy as the respective sides. The hypothenuse will be the distance yc_1, laid off in either direction from ab.

2. The angle abc_1, will be the true angle between the lines.

Notes.—1. A problem like this requires no *GL*.

2. This problem can be done in the same manner if one of the lines is parallel to *V* or *P*.

3. This problem can be done exactly as in Problem 14, but the method here given is an easier solution for this situation of the lines.

4. It may help the student to transfer *GL* to *a'b'*, thus putting *AB* into *H*, without changing the relation of the lines.

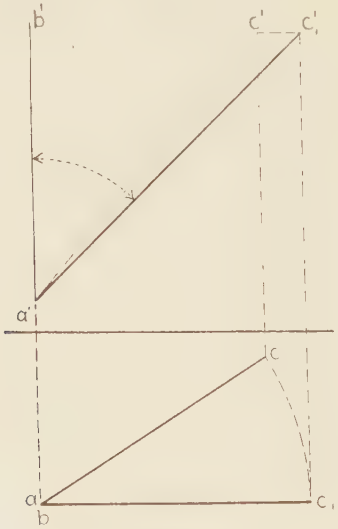


Fig. 28.

57. Problem 16. To measure an angle, when one side is perpendicular to H.

Analysis.—Revolve the oblique line about the perpendicular line as an axis, until it is parallel to *V*.

Construction.—Use the method employed in solving Problems 10 and 11 (see Fig. 28). In this figure AB is the perpendicular to *H*, CB the oblique line, and ABC the angle to be measured. Rotating BC until bc is parallel to *GL*, we obtain the angle $a'b'c'_1$, which is the required angle.

58. Problem 17.—To find the projection of the bisector of any angle.

Analysis.—1. Revolve the lines to show the true angle, by the method shown in any of the three preceding problems, as the case may require.

2. Bisect the angle in its revolved position.

3. Revolve any point of the bisector back into the original plane of the lines.

4. Connect this point with the vertex.

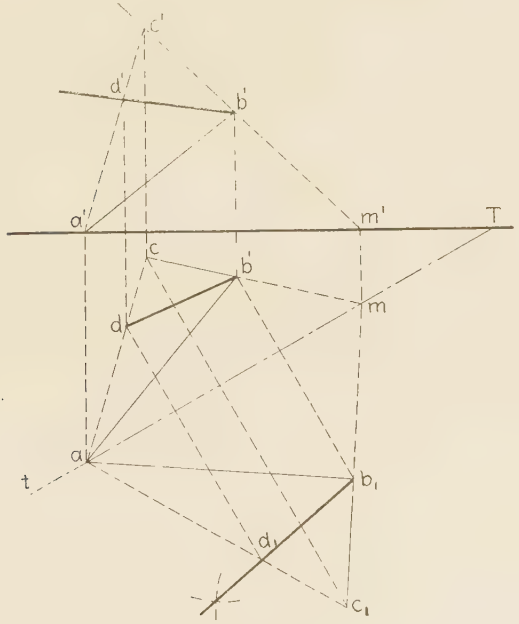


Fig. 29.

Construction.—It is required to find the projections of the bisector of the angle ABC, Fig. 29.

1. Obtain the trace tt.

2. Revolve B to b_1, and C to c_1, and connect a, b_1, and c_1 to obtain the true angle.

3. Draw the bisector, d_1b_1, of the angle ab_1c_1.

4. Draw the auxiliary line ac_1, intersecting b_1d_1 in d_1.

5. Revolve ac_1 back to its original position, ac,a′c′.

6. Revolve d_1 back to d on ac.

7. Project d to d′ on a′c′.

8. Draw b′d′ and bd, the required projections of the bisector.

Note.—When the angle to be bisected is included between the piercing points and the vertex, as is the angle ABM in Fig. 29, the auxiliary line is unnecessary. This is so, because the bisector will intersect the trace, and the return revolution can be effected without difficulty. Give the reason.

Why do not the projections of the bisector of an angle bisect the corresponding projections of the angle?

EXERCISES

59. Measure the angles included between the following lines.

118. A(2+½−1) B(3+1½−2¼) and BC(4+¼−¼).
119. D(1¾+⅝−2½) E(4+1½−1) and EF(2¼+2½−½).
120. G(1+2−2) H(4+1−1¼) and HK(1½+1¼−¾).
121. M(2+2−2) N(2+1−½) and NO(4+2−2).
122. A(2+2−½) B(2+½−2) and BC(4+2−½).
123. D(1+2−1) E(2¼−1+1½) and EF(3+1+2).
124. G(1½, 0, 0) H(3+1−1½) and HK(4¾, 0, 0).
125. M(2+2−2) N(4+1−½) and NO(2+1¼−1) by revolving into the profile.
126. A(1½+2+2) B(3½+1+1) and BC(2+½+¼).
127. D(2−½−½) E(3¼−¾−¾) and EF(2½−1½−1½).
128. G(2+1½−2) H(4+1½−1) and HK(2½+1−¼).
129. M(5+1¾−1¼) N(2½+1¾−2) and NO(4½+½−½).
130. A(2+2−1½) B(4+2−1½) and BC(2+1−1).
131. D(1½+1−¾) E(4+1½−¾) and EF(2+½−2).
132. G(2−1½+1) H(4−1½+2) and GK(4−1+½).
133. M(1½+1½−½) N(1½+1½−2) and NO(4+½−1).
134. A(2, 0, 0) B(4+2−1½) with GL.

Find the angle between the traces of the following planes.

135. T(5+2) 1½(5−1½).
136. S(1+2) 4½(1+1).
137. R(1+2) 3½(5−2).
138. W(1+2) 3½(5−1).
139. T(2½+2) 2½(1−1), and of U(5+2) 3(3−2).

Measure the angles, and find the true size of the following set of triangles.

140. A(1+½−¼) B(2¼+2−2½) C(4+2−1).
141. E(2−½+½) F(2+1¼−2) G(3½+1½−1½).
142. H(2+2½−2) K(3+2−1) L(4+1¼−1).
143. Measure the angles and true size of the parallelogram A(2+1½−1) B(3+2¼−⅝) C(4+1¼−1½) D(3+½−⅞).
144. Prove graphically that D(2+1¼−2) E(3½+1¼−½) F(5+½−2) is a right angle. Write an explanation.
145. Draw a line from O(2¾+2−1½) to intersect M(1+½−¾) N(3½+1−1½) at 45°.
146. Measure the angles between two adjacent edges, and between an edge and a base line of the square pyramid given in Exercise 113, Art. 54.

147. Measure the angle that H(2+1−½) K(4+2½−1) makes with its *H*-projection. Also that which it makes with its *V*-projection.
148. Draw the roof shown in Fig. 25, and measure the following angles therein:
(a) Hip rafter EF with hip rafter EG.
(b) Hip rafter EF with eave FG.
(c) Hip rafter AB with eave AG.
(d) Hip rafter AB with ridge BE.
149. Draw the roof shown in Fig. 25 to a definite scale, and measure the true size of the roof planes C and D.
Compute their areas from the scale.
What would it cost to cover the roof with material costing $1.50 per square yard?
150. The *H*-trace of T(x, x) 4(2−1½) makes 75° with its *V*-trace. Draw the *V*-trace.
151. A ray of light from P(1¼+1−1½) is reflected at A in *V* to B in *H*, and thence to O(5+2½−2½). Locate A and B.
152. Find the center of the triangle C(1+1+2) D(2½−2−1) E(4+½+1½).
153. Draw a line through O(2+1, x) in the plane T(1½+3) 4½(1½−2) making 45° with the *H*-trace.

Find the projections of the bisectors of the following angles.

154. A(1+½−⅜) B(2½+2−2) C(4+1−½).
155. D(2+2−1¾) E(4+1¼−1¼) F(2½+1−½).
156. G(1½+2−2) H(4+¾−¾) K(1½+1¼−1¼).
157. M(2+2+2) N(4+1+1) O(2½+½+½).
158. A(2+1½−2) B(3½+1½−¾) C(2½+1−¼).
159. D(2+¾−1) E(4+¾−1) F(3+2−½).
160. G(2+2−2) H(2+1−½) K(3½+1¾−2).
161. M(2½+1½−½) N(2½+1½−2) O(4+½−1).
162. A(2+1½−2) B(4, 0, 0) C(1, 0, 0).
163. Between the *H*- and *V*-traces of T(1½+2) 4(1½−1½).
164. Between the *H*- and *P*-traces of S(4½+1½) 1(4½−2).
165. Between the *H*- and *V*-traces of R(2+2) 4(2+1).
166. Between the *H*- and *V*-traces of X(1+2) 3(5−2).
167. Between the *V*- and *P*-traces of W(+1½) ∞(−1).
168. D(2+1½−1½) E(4+½, 0) and its *V*-projection.
169. F(3+2−½) G(3+1−2) H(3+½−¾).
170. M(2½−½+½) N(2½−2+2) O(2½−1+1½).

60. Problem 18.—To find the shortest distance from a point to a line.

From Plane Geometry we know that the shortest distance from a point to a line is on the perpendicular through the point to the line.

Analysis.—1. Pass a plane through the point and the line.

2. Revolve the point and line about either trace of the plane to show their true relation.

3. From the revolved position of the point erect a perpendicular to the revolved position of the line.

4. The length of this perpendicular is the shortest distance.

Construction.—Let AB (Fig. 30) be the given line, and C the given point.

1. Draw the *H*-trace of the plane containing the line and point.

Note.—Either trace may be used.

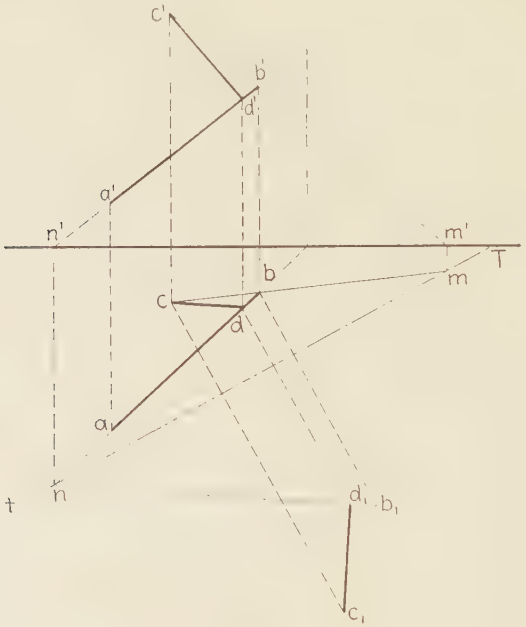


Fig. 30.

2. Revolve AB to a_1b_1.

3. Revolve C to c_1.

4. Draw c_1d_1 perpendicular to a_1b_1. This is the required distance.

61. Problem 19. To find the projections of any given plane figure lying in an oblique plane.

This problem will be found to use the principles employed in the three preceding problems, but with the order of operation partly reversed.

Analysis.—1. If no point or line of the required figure is given, assume such a point or line in the given plane.

2. Revolve the point or line thus assumed about the H-trace into H, or about the V-trace into V.

3. Construct the figure in its true size on or about the revolved point or line.

4. Revolve the figure back into the plane in the original position of the plane.

62. Counter-revolution of revolved points and lines.

Return-revolution, or counter-revolution, as it is also called, is the only point of this construction that is new or difficult. Therefore, before taking up the construction, let us consider it. We have had a specimen of it in finding the projections of the bisector of an angle, Problem 17. Inasmuch as this is, perhaps, the most generally practical application of Descriptive Geometry, it would be well to do many problems and exercises involving its use. Two methods of return-revolution will be given.

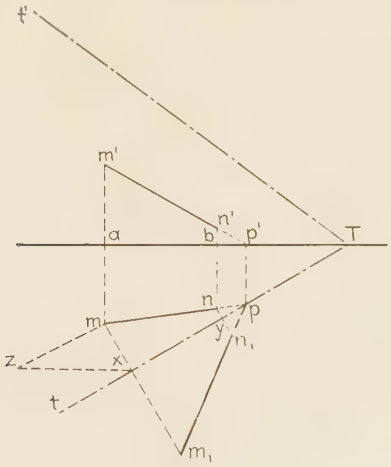


Fig. 31.

In the revolution of any line about the trace of its plane, two facts should be borne in mind; **The piercing point of the line does not move in revolution,** because, as we have seen, it is in the trace; that is, its axis. Also, **if the line is parallel to the trace, it will be parallel when revolved,** and the converse of this is true. With these two rules, any problem in counter-revolution may be worked.

Let M (Fig. 31) be a known point in the plane T. Revolve M about the H-trace to m_1. Through m_1 draw any line m_1n_1.

It is desired to counter-revolve the line m_1n_1 into the plane T.

It is obvious that m_1 will return to mm′; that is, where it came from.

If m_1n_1 is parallel to Tt, it will be parallel when it is returned to T, and its *H*-projection can be drawn through m parallel to the *H*-trace. The point n_1 will of course return in a circle perpendicular to the axis.

If m_1n_1 is not parallel to Tt, mn will not be parallel to Tt, and both will therefore have to intersect Tt. From what we already know of revolution, we know these intersections must be in one point.

Extend m_1n_1 to Tt at the point p. The line m_1p will contain the point n_1, so n will lie on mp. We know the actual positions in space of M and P, and as N must lie in that line, its projections must be on the projections mp, m′p′. Revolve n_1 on a perpendicular to the trace to the line mp, and then project to m′p′, and we have the line MN revolved back into the plane.

This is the more convenient method of counter-revolution, but some may find the work easier understood by the Similar Triangles Method.

SIMILAR TRIANGLES METHOD

This method depends on the fact that all points in a plane, when rotating about one of the traces, have for radii the hypothenuses of similar triangles.

Referring to Fig. 31, m_1x and n_1y are the radii of the revolution of the points M and N about Tt. They are, therefore, the hypothenuses of triangles, whose bases and altitudes are respectively mx, ny, m′a, and n′b, and the triangles are similar. So, to revolve a point, as *N*, back to the original position of the point in the plane, 1. Draw the path of revolution through n_1 perpendicular to the trace, intersecting it at y.

2. The perpendicular n_1y will be the hypothenuse of the triangle to be formed.

3. With yn_1 as a hypothenuse, construct a triangle similar to mzx.

4. The sides will be respectively equal to ny and n′b.

5. Lay off ny on the path of revolution, locating n.

6. Project n to n′, locating it in the plane by any of the regular methods.

Construction.—Let it be required to draw the projections of a 1-in. square in T (Fig. 32) about the point O (in T) as a center. Let it be stipulated that two sides of the square be parallel to *H*.

1. Locate O in T by Problem 5.

2. Revolve O about Tt into *H* at o_1 by Problem 9.

3. Construct the 1-in. square $a_1b_1c_1d_1$ about o_1 as a center, in its actual size. As two sides are specified parallel to *H*, we draw c_1d_1 and a_1b_1 parallel to Tt. See Theorems XXVII and XXVIII.

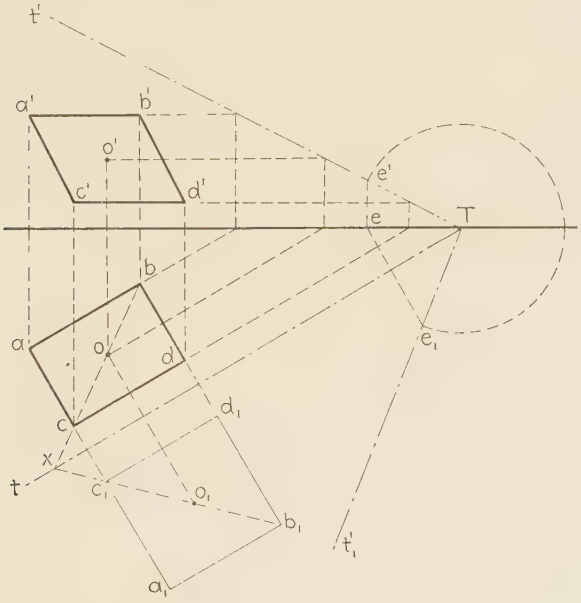


Fig. 32.—A square drawn in an oblique plane.

4. Draw the auxiliary line b_1x through b_1, o_1, and c_1.

5. Draw xo, prolonging it as much as necessary.

6. Revolve b_1 and c_1 about Tt to the points b and c on the line xo.

7. Draw ab and cd parallel to Tt, and complete the parallelogram abcd.

8. Project a, b, c and d to their *V*-projections, locating them in T, at the points a,′ b′, c′ and d′.

This gives us both projections of the square.

Notes.—1. The points might have been rotated back into T by the similar triangles method.

2. It is sometimes desirable to show the V-trace revolved about the H-trace into H. It is desirable in order to limit the figure to one space angle. By taking E (Fig. 32) any point on the V-trace, and revolving it about the H-trace into H at e_1, the revolved trace may be drawn through T and e_1. Note that Te′ and Te_1 are equal, and that the revolution can be made by striking an arc with Te′ as a radius until it intersects the perpendicular from e to the trace at e_1.

63. EXERCISES

Find the shortest distance from the following points to the following lines:

171. A($2\frac{1}{2}+2-2$) to B($2+\frac{1}{2}-1$) C($4+2$, 0).
172. D($3+1\frac{1}{2}-2$) to E($2+1-\frac{1}{2}$) F($4+1-\frac{1}{2}$).
173. G($3+1\frac{1}{2}-2$) to GL.
174. H($3-1-1$) to K($4+2+2$) L($4+1+1$).
175. M($2+1$, x) in T($2+3$) $4\frac{1}{2}(2-2)$ to the H-trace, and to the V-trace.
176. N($2\frac{1}{2}$, x, 0) on the V-trace of T($2+3$) $4\frac{1}{2}(2-2)$ to the H-trace.
177. A 3-in. steam pipe runs through a basement from A(3 ft., 0 − 9 ft.) to B(15 ft. + 7 ft. 6 in., 0). It is desired to run the shortest possible $1\frac{1}{2}$-in. pipe from a wall connection at C(5 ft. 3 in., + 7 ft. 6 in., 0) to the main pipe. Allowing 3 in. for connections, what will be the length of the smaller pipe? Scale 1 in. = 10 ft.

Draw the projections of the following plane figures according to the conditions given.

178. A $1\frac{1}{2}$-in. square in T($1+3$) $4(1-2)$, one side || to H, and $\frac{1}{2}$ in. from the H-trace.
179. A $1\frac{1}{2}$-in. square in T($1+3$) $4(1-2)$, center at O($2+1$, x), its diagonal || to V.
180. A right triangle, base $1\frac{1}{4}$ in., altitude $\frac{3}{4}$ in. in T($1+3$) $4(1-2)$. Center at O($2+1$, x) in T.
181. An equilateral triangle, $1\frac{1}{4}$-in. sides, in T($1+3$) $4(1-2)$. Center at O($2+1$, x) in T.
182. A regular hexagon of 1-in. sides, in T($1+3$) $4(1-2)$. Center at O($2+1$, x) in T.
183. A regular pentagon of 1-in. sides in T($1+3$) $4(1-2)$. Center at O($2+1$, x) in T.
184. A $1\frac{1}{2}$-in. circle in T($1+3$) $4(1-2)$. Center at O($2+1$, x) in T.

Note.—Both projections will be ellipses, and each major axis will be parallel to the respective trace. Find the minor axis of each ellipse, which will be the diameter of the circle perpendicular to the respective trace. Having these two dimensions of each ellipse, draw the curves by any of the usual methods, preferably the "trammel method." Of course there can be no objection to dividing the circle into any number of parts, and counter-revolving the points thus obtained, but it is more laborious. It should, of course, be noted that the two minor axes are not projections of the same line.

185. Inscribe a circle in the triangle, Exercise 181.
186. Draw a 1-in. circle in the center of the hexagon in Exercise 182.

187. A 1-in. square in S(+2)∞(−1½), one side || to *H* and ⅝ in. from the *H*-trace.
188. A 1¼-in. square in S(+2)∞(−1½), the diagonal || to *H*, center at M(3+¾, x) in S.
189. A right triangle, base 1¼ in., altitude ¾ in. in S(+1½)∞(−1¾). Draw its hypothenuse || to *H*, ½ in. from the *H*-trace.
190. An equilateral triangle, 1¼-in. sides, in S(+2)∞(−1½), whose base is in the line, which is the locus of all points in S that are ½ in. above *H*.
191. A regular hexagon, 1½ in. across the flats, in S(+2)∞(−1½). Center at K(3+¾, x) in S.
192. A regular octagon, ¾-in. sides, in S(+2)∞(−1½). Center at O(3+¾, x) in S.
193. A 1½-in., long diameter, regular hexagon with a ¾-in. circle in the center in S(+2)∞(−1½). Center at M(3+1, x) in S.
194. Inscribe a circle in the triangle in Exercise 190.
195. A 1-in. square in R(1+2)3 (5−2), || to *H*, center at A(2½+1½, x) in R.
196. A 1¼-in. square in R(1+2) 3(5−2), its diagonal || to *V*, 1 in. from the *V*-trace.
197. A right triangle, whose base is 1¼ in., altitude ¾ in., in R(1+2) 3(5−2). Draw the hypothenuse || to *H*, ¾ in. from the *H*-trace.
198. An equilateral triangle in R(1+2) 3(5−2), whose base is the line A(2+2, x) B(3+1, y) in R.
199. A regular hexagon, 1½ in. long diameter, in R(1+2) 3(5−2), its center at O(2½+1½, x) in R.
200. A 1½-in. circle in R(1+2) 3(5−2), center at M(2½+1½, x) in R.
201. A 2-in. octagon in R(1+2) 3(5−2), two sides || to H, center at M(2½+1½, x) in R.
202. A 1½-in. square in Q(3+3) 3(5−2), diagonal || to *H*, center 1½ in. above *H* and 1 in. from *V*.
203. A 1½-in. square in Q(3+3) 3(5−2), two sides || to *H*, center at M(4+1, x) in Q.
204. A right triangle, base 1¼ in., altitude 1 in., in Q(3+3) 3(5−2), hypothenuse || to *V*, ½ in. from *V*.
205. An equilateral triangle in Q(3+3) 3(5−2), its base A(4+1½, x) B(5+½, x) in Q.
206. A regular hexagon, 1-in. sides, in Q(3+3) 3(5−2), center at M(4+1, x) in Q.
207. A regular hexagon, 1½-in. short diameter, in Q(3+3) 3(5−2), with its diagonal on the line E(3+1, 0) F(5, 0−2). Draw a 1-in. circle in its center.
208. Inscribe a circle in the triangle in Exercise 205.
209. Circumscribe a circle about the hexagon in Exercise 206.
210. A 1-in. square in W(1+3) 3(5−1½), with one side in the line C(3½+1, x) D(4½+½, x) in W.
211. A square in W(1+3) 3(5−1½), whose diagonal is E(3½+1¼, x) F(4½+⅜, y) in W.
212. A 1¼-in. square in W(1+3) 3(5−1½) with one corner touching *H*, and a side making 30° with the *H*-trace.
213. A 1¼-in. square in W(1+3) 3(5−1½), having two sides parallel to *H*,

one of them $\frac{1}{4}$ in. from the H-trace, the entire square being within the first angle.

214. A $1\frac{1}{2}$-in. equilateral triangle in W$(1+3)$ $3(5-1\frac{1}{2})$, center at A$(4\frac{1}{2}+\frac{3}{4}, x)$, with one corner touching H.

215. Inscribe a circle in the triangle in Exercise 214.

216. A $1\frac{1}{2}$-in. hexagon in W$(1+3)$ $3(5-1\frac{1}{2})$, with two sides parallel to H. Center at M$(4+\frac{3}{4}, x)$ in W.

217. An octagon, sides $\frac{5}{8}$ in., in W$(1+3)$ $3(5-1\frac{1}{2})$, whose center is M$(4+1, x)$ in W.

218. A circle in W$(1+3)$ $3(5-1\frac{1}{2})$, one of whose diameters is the line K$(3\frac{1}{2}+1\frac{1}{2}, x)$ L$(4\frac{1}{2}+\frac{3}{4}, y)$ in W.

219. The roof of a tower is a hexagonal pyramid, whose base is 12 ft., long diameter, and whose altitude is 15 ft. In each of the three front sides are rectangular windows 3 ft. by 5 ft., each 3 ft. from the base of the tower. Draw the projections. Scale $\frac{1}{8}$ in. $= 1$ ft.

64. Problem 20.—To find the line of intersection of two planes.

Axioms.—1. The intersection of two planes is a straight line.

2. Two points determine a straight line.

Analysis.—1. Find both projections of the point where the V-traces of the given planes intersect.

2. Find both projections of the point where the H-traces intersect.

3. Join the respective projections of these two points, which are common to both planes.

4. The line thus drawn is common to both planes, and therefore their intersection.

Note.—The H-projection of any V-trace is in GL, and therefore the H-projection of any point in the V-trace is in GL.

Proof.—The V-traces, being each a line in one of the given planes, intersect in a point common to both planes, and therefore in the line of intersection. Similarly for the intersection of the H-traces. Having found these two points, the line of intersection is determined.

Construction.—Left to the student.

65. Problem 21. To solve the foregoing problem when the traces do not intersect within the limits of the problem, or not at all.

Analysis. 1. If one pair of traces intersect, locate their point of intersection.

2. Draw an auxiliary plane that will intersect both given planes. The lines of intersection of the auxiliary and given planes will intersect in a point on the required line of intersection.

3. **If neither pair of traces intersect,** draw a second auxiliary plane which will determine another point on the required line, thus determining the line of intersection.

Construction.—For most oblique planes, such as the two shown in Fig. 33, it is quickest and most convenient to use planes parallel to H or V as auxiliaries. Such planes have one trace only, and that is parallel to GL. Such planes intersect oblique

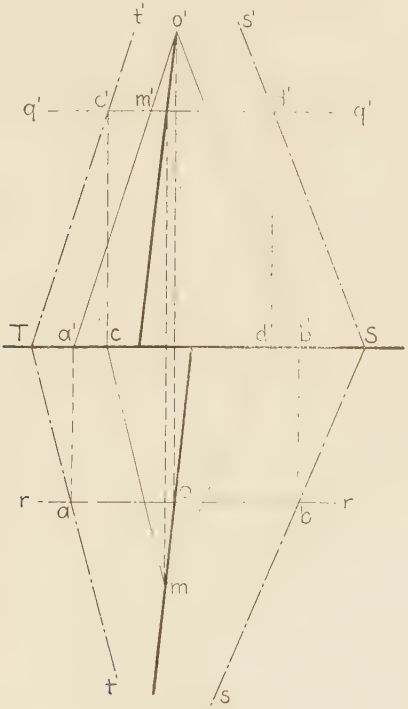


Fig. 33.

planes in lines parallel to their traces; that is, a plane parallel to V will intersect an oblique plane in a line parallel to its V-trace, etc. Let the student give the reason from Solid Geometry.

1. Draw q'q', the V-trace of Q, parallel to H. The lines cut by Q from T and S are CM and DM.

2. Locate m and m', their intersection.

3. Draw rr, the H-trace of R, parallel to V. The lines cut by R from T and S are AO and BO.

4. Locate o and o′, their intersection.

5. Draw o′m′ and om, the required line of intersection.

66. Special Case 1.— It sometimes happens that only one point on the line of intersection can be conveniently located by the auxiliary planes just shown. Fig. 33*a* gives such a case. In a case like this an auxiliary plane parallel to one of the planes may be used to determine the **direction** of the line of intersection. This is true, because two parallel planes cut a third plane in parallel lines. Fig. 33*a* shows two planes, T and S, in which an auxiliary plane Q, parallel to S, is used.

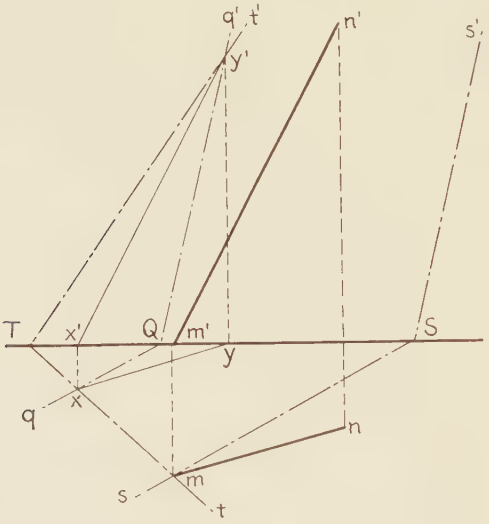


Fig. 33*a*.

Q intersects T in the line XY. The *H*-traces of S and T intersect in the point M. Through M draw MN parallel to XY. MN will be the intersection of S and T.

Special Case 2.— When both planes are parallel to GL.

In this case the traces are all parallel, yet the planes may intersect. The line of intersection will be **parallel to** *GL*, because both planes are. Hence, one point on the line of intersection is all that is necessary to determine the line.

Figs. 34 and 35 show two different auxiliary planes that may be used. In Fig. 34 a profile plane is used, and the profile traces yield the intersecting point.

In Fig. 35 an oblique plane, Q, is used, which intersects T in the line CD, and S in the line EF. The lines CD and EF intersect in the point G, through which the line of intersection, AB, may be drawn.

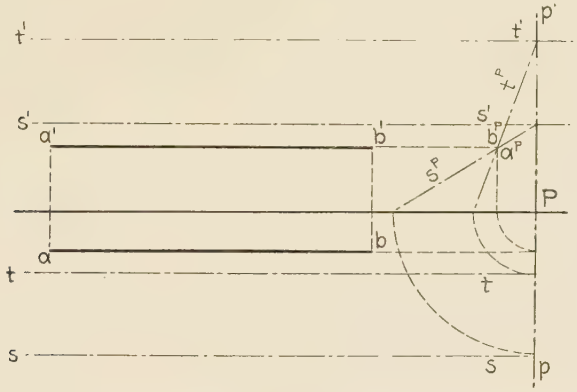


Fig. 34.

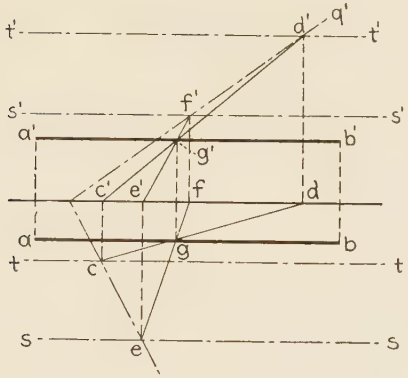


Fig. 35.

67. Note.—The foregoing solutions give an idea of the multitude of resources open to the student in the solving of varieties of problems. Sometimes one method or auxiliary plane will be useful, which would be awkward under other conditions. No stereotyped method of performing these solutions can be given, for each problem will bring up conditions, where there is a preference of methods. The analyses are exactly alike, and they must be learned, but not memorized.

EXERCISES

68. Find the intersections of the following planes.

220. T(2+2) $4\frac{1}{2}(2-2\frac{1}{2})$ and S(5+$1\frac{1}{2}$) 3(5−2).
221. R($1\frac{1}{2}$+3) 4(1−1) and Q($3\frac{1}{2}$+3) 3(5−$\frac{3}{4}$).
222. X(1+3) 4(5−2) and Y(5+3) 2(1−3).
223. W(2+3) 1(5−3) and Z(1+3) 4(4−3).
224. T(1+$2\frac{1}{2}$) $4\frac{1}{2}$(1−3) and S(2+3) 2(5−2).
225. R(5+$2\frac{1}{2}$) 2(5−3) and M, parallel to *H*, 1 in. above. Also $\dot{\text{R}}$ with Q, parallel to *V*, 1 in. behind.
226. T (+$1\frac{1}{4}$)∞(−$\frac{3}{4}$) and S(2+3) 4(2−3).
227. X($1\frac{1}{2}$+$2\frac{1}{2}$) 3(5−2) and Y(+2)∞(−$1\frac{1}{4}$).
228. W(5+3) 2(5−2) and a plane containing the point A($3\frac{1}{2}$+$2\frac{1}{2}$−$2\frac{1}{2}$) and *GL*.
229. R(2+$2\frac{1}{2}$) 4(2−3) and Q(−$\frac{1}{2}$)∞(−2).
230. T(1+2) 4($5\frac{1}{2}$−3) and S(5+$2\frac{1}{2}$) $2\frac{1}{2}$(1−2).
231. X($1\frac{1}{2}$+3) 4($5\frac{1}{2}$−3) and Y($4\frac{1}{2}$+3) 2($4\frac{1}{2}$−3).
232. W(2+3) $\frac{1}{2}$(2−3) and Z($3\frac{1}{2}$+3) $5\frac{1}{4}$(4−3).
233. Q($\frac{1}{2}$+3) 2(4−2) and R(2+3) $2\frac{3}{4}$($5\frac{1}{2}$−2).
234. T(1+1) 4(1−$1\frac{1}{2}$) and S(2+3) $4\frac{1}{2}$($5\frac{1}{2}$−1).
235. X(1+3) 3(5−3) and Y(5+2) 2($\frac{1}{2}$−1).
236. W(1+3) ($4\frac{1}{4}$+2) (1−$1\frac{5}{8}$) (3−$1\frac{1}{4}$) and Z(1+$2\frac{3}{4}$) (6+$\frac{5}{8}$) ($3\frac{7}{8}$−3) ($5\frac{1}{4}$−$\frac{7}{8}$).
237. Q(1+2) $3\frac{1}{2}$(1−3) and R($4\frac{1}{2}$+2) $3\frac{1}{2}$(5−1).
238. S(5+$2\frac{1}{4}$) $3\frac{1}{4}$($1\frac{1}{2}$−$2\frac{1}{4}$) and T(1+2) $3\frac{1}{4}$($5\frac{1}{2}$−2).
239. W(+2)∞(−$\frac{1}{2}$) and X(+$1\frac{1}{4}$)∞(−$1\frac{1}{4}$).
240. Y(+$1\frac{1}{2}$)∞(−2) and Z(−$\frac{1}{2}$)∞(−$\frac{3}{4}$).
241. Q(−$\frac{3}{4}$)∞(−$\frac{1}{2}$) and R(−$1\frac{1}{2}$)∞(+1).
242. S(+2)∞(+1) and T(+$1\frac{1}{4}$)∞(+$2\frac{1}{2}$).
243. W(+2)∞(−$\frac{3}{4}$) and X(−$\frac{1}{2}$)∞(+2).
244. Draw a line AB parallel to S(2+2) $4\frac{1}{2}$(2−$2\frac{1}{2}$) and T(5+$1\frac{1}{2}$) $2\frac{3}{4}$(5−$2\frac{1}{2}$) through the point A(4+2−2).
245. Find the point of intersection of the three planes X($4\frac{1}{2}$+2) 1($4\frac{1}{2}$−3), Y(2+3) 5(3−3) and Z(+$\frac{1}{2}$)∞(−$1\frac{1}{2}$).
246. Find the point of intersection of the three planes, T(3+3) 1(4−3), S(1+1) 5(6−1) and R(2−2) 4(6+2).
247. Find the projections of the tetrahedron included between the planes X(1+2) 2(5−2), Y(5+2) $4\frac{1}{2}$(1−2), Z(+$2\frac{1}{2}$)∞(−2) and *H*.
248. Develop the surface of the tetrahedron in Exercise 247.
249. Find the intersection of the plane T(1−1) $5\frac{1}{2}$(3−3) and the tetrahedron in Exercise 247.
250. Draw the projections of an equilateral triangular prism, of $\frac{1}{2}$-in. sides, whose ridge is the line A(1+$\frac{3}{4}$−$1\frac{1}{2}$) B($3\frac{1}{2}$+$\frac{3}{4}$−$\frac{1}{2}$), and of a cube resting on *H*, whose base diagonal is the line C(2+0−1) D($3\frac{1}{2}$, 0−2). Find the line of intersection of cube and prism without finding the traces of their bounding planes.
251. Draw the projections of a pyramid A(2, 0−$1\frac{3}{8}$) B(3, 0−$\frac{5}{8}$) C($3\frac{3}{4}$, 0−$1\frac{5}{8}$) D($2\frac{3}{4}$, 0−$2\frac{3}{8}$) E($2\frac{7}{8}$+$1\frac{1}{2}$−$1\frac{1}{2}$) and the triangular prism of $\frac{3}{4}$-in. sides, one side parallel to *H*, whose center line is K($1\frac{3}{4}$+$\frac{3}{4}$−1) L($4\frac{1}{4}$+$\frac{3}{4}$−$1\frac{1}{2}$). Draw their line of intersection.

69. Problem 22.—To find the piercing point of a line and plane.

Analysis.—1. Pass a plane through the given line.

2. Find the line of intersection between the given plane and the auxiliary plane.

3. Where this intersection crosses the given line is the required piercing point.

Note.—Memorize this analysis. It is one of the most important in the work.

Proof.—Since the auxiliary plane contains both the given line and the intersection with the given plane, the two lines

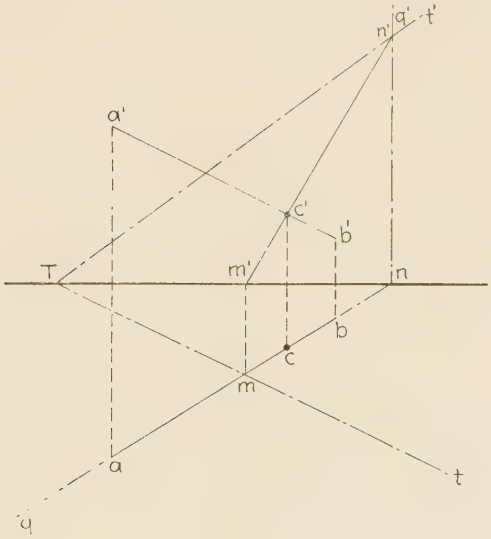


Fig. 36.

must intersect, or be parallel. If they are parallel, the given line is parallel to the given plane, and there is no piercing point.

Construction.—Let AB (Fig. 36) be the given line, and T be the given plane.

1. Pass a plane (Q) through AB.

Note.—In nearly all cases the simplest plane to draw is a projecting plane. In Fig. 36 the plane q'Qq is the *H*-projecting plane of AB. Any plane containing the line may be used, but the projecting plane is almost always the easiest.

2. Find MN, the intersection of T and Q.

3. MN crosses AB at C, and therefore C is the required piercing point of AB and T.

70. Problem 23.—To find the piercing point of a line and plane without obtaining the traces of the plane.

Analysis.—Let the plane be determined by two intersecting, or two parallel lines.

1. Pass a projecting plane through the line.

2. Find their piercing points with this plane.

Note.—This operation requires no auxiliary planes or lines, as will be seen in the construction.

3. Join these piercing points by a line.

4. Where the line, or its extension, crosses the given line will be the required piercing point.

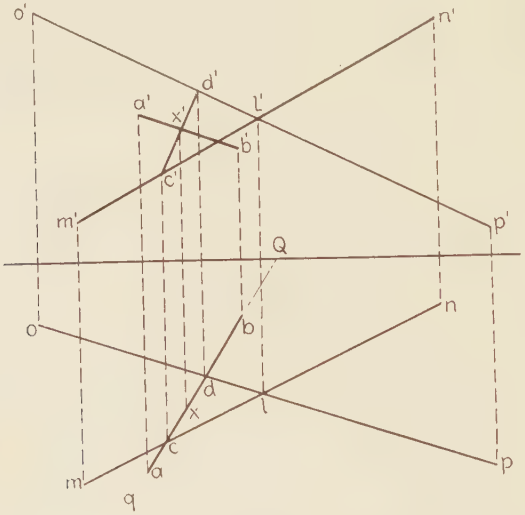


Fig. 37.

Construction.—Let AB, Fig. 37, be the given line. Required to find its piercing point with the plane of MN and OP.

1. Draw the *H*-trace, Qq, of the *H*-projecting plane of AB.

Note.—One trace only is necessary.

2. The points in which this auxiliary plane crosses the projections op and mn will be the *H*-projections, c and d, of the piercing points of OP and MN with Q.

3. Erect projectors from c and d to c′ and d′ on m′n′ and o′p′, and join c′d′.

4. Where c′d′ crosses a′b′ at x′ is the *V*-projection of the required piercing point.

5. Project x′ to x on ab for the required *H*-projection.

Note.—The profile plane cannot often be used in these problems, although many students try to use it on every occasion. It is only necessary or available when dealing with a line whose projections are perpendicular to *GL*, and sometimes when dealing with a plane parallel to *GL*.

EXERCISES

71. Find the piercing points of the following lines and planes.

252. A(2+¾−1) B(4+3−2) and T(1+3) 4½(1−2½).
253. C(5+2−½) D(3½+½−2) and S(2+2) 5(2−3).
254. E(3+2−2) F(3+1−½) and R(1+3) 4½(1−2½).
255. G(2+1−1) K(4½+1−2½) and Q(4+3) 3(4−2½).
256. M(2+1¼−¾) N(4+1¼−¾) and W(1+3) 4½(1−2½).
257. O(1½+2−2) P(4+½−½) and Y(2+3) 2(5−2½).
258. A(3+½−2) B(5+2−½) and T(1+3) 2½(5−1¼).
259. C(3+1−2) D(4½+2−1) and S(5¼+1½) 3(¾−1½).
260. E(5+1½−1¾) F(4+½−1¼) and R(5+3) 2(5−1½).
261. G(1½+2−2) K(3½+1−½) and Q(+1¼)∞(−¾).
262. M(3+2−2) N(3+1−½) and W(+1½)∞(−1).
263. A(2+1½−¾) B(4+1½−¾) and C(1½+1½+1½) D(3½−1¾−1¾) and the plane X(5+2) 3(1−2). Find the distance between the piercing points of AB and CD with X.
264. E(2+½+½) F(4+2+2) and Y(1+3) 4½(1−2).
265. M(2½+2½−2) N(4+½−½) and Z(1+2) 3(5−1¼).

Find the piercing points of the following lines and the planes of the given lines, without finding the traces of the planes.

266. A(4+1−1¼) B(5½+2−1¾) with the plane of C(1¼+¼−1¾) D(2½+1−½) and DE(3−⅜−⅝).
267. G(1¾+3¼−2½) H(2¾+1¾−1¼) with the plane of K(1½+1¾−½) L(2½+1−1) and LO(2+¼−1½).
268. A(3+2−2) B(3+½−1) with the plane of C(3½+2−½) D(3½+½−2) and E(4½+2−½) F(4½+½−2).
269. From the middle point, O, of the line G(3½+¼−1) H(4¾+1−1¾) draw a line which will intersect both K(½+1⅛−3) L(1½+3−2⅝) and M(2+1½−2½) N(3+2−¼).
270. Erect a square pyramid of 2-in. altitude, on *H*, having for its base diagonal the line A(2¼, 0−¾) B(4, 0−1½). Find where the line C(4½+1¾−1) D(1½+½−2) pierces the pyramid, entering and leaving.
271. Find the shortest line that can be drawn on the surface of the pyramid, in Exercise 270, between the two piercing points of the line CD in the same Exercise.
272. Pass the plane T(1+1½) 5(3−3) through the pyramid in Exercise 270. Crosshatch the section, and show it in its true size also.

273. Develop the surface between H and T of the pyramid as found in Exercise 272. Make a paper model of it.

274. Find the true length of that part of the line A$(1+2-2\frac{1}{2})$ B$(4+\frac{1}{2}-\frac{1}{2})$ included between the planes T$(3+2)$ $1\frac{1}{2}(3-1\frac{1}{2})$ and S$(5+3)$ $3(5-3)$.

275. Find the true length of that part of the line C$(2+\frac{1}{2}-\frac{1}{2})$ D$(4+2\frac{1}{2}-1\frac{1}{2})$ included between the planes X$(+2\frac{1}{4})\infty(-1\frac{3}{4})$ and Y$(+1\frac{1}{4})\infty(-1\frac{1}{4})$.

72. Problem 24.—To draw a line through a point perpendicular to a plane.

Analysis.—According to Theorem XXIV, Art. 25, a line that is perpendicular to a plane must have its projections perpendicular to the respective traces of the plane; therefore,

1. From the projections of the point draw projections of the line perpendicular to the respective traces of the plane, H to H, and V to V.

Let the student make the construction.

73. Problem 25.—To draw a perpendicular of any given length from a point in a plane.

Analysis.—1. Erect a perpendicular of indefinite length from the given point, according to Problem 24.

2. Assume a point on the perpendicular.

3. Revolve the definite line, thus drawn, so as to show its true length.

4. Lay off the required distance on the true length of the line from the given point.

5. Return the limiting point to its proper position in space. Compare this with Problem 13.

Construction.—Let it be required to erect a perpendicular 1 in. long at the point A in the plane T, Fig. 38.

1. Draw ab and a′b′, respectively perpendicular to t′T and Tt.
2. Assume b′b, anywhere on the perpendicular.
3. Measure AB, true length $a'b'_1$.
4. Lay off $a'c'_1$ on $a'b'_1$, 1 in. from a′.
5. Return c'_1 to c′ on a′b′.
6. Project to c on ab.

AC is then perpendicular to T and 1 in. long.

74. Problem 26. To draw the projections of a solid with its base on a given oblique plane.

Analysis.—1. Lay out the base according to Problem 19.

If it be a prism,

2. Erect perpendiculars to the plane of the base of the specified altitude from each of the corners of the base, according to Problem 25.

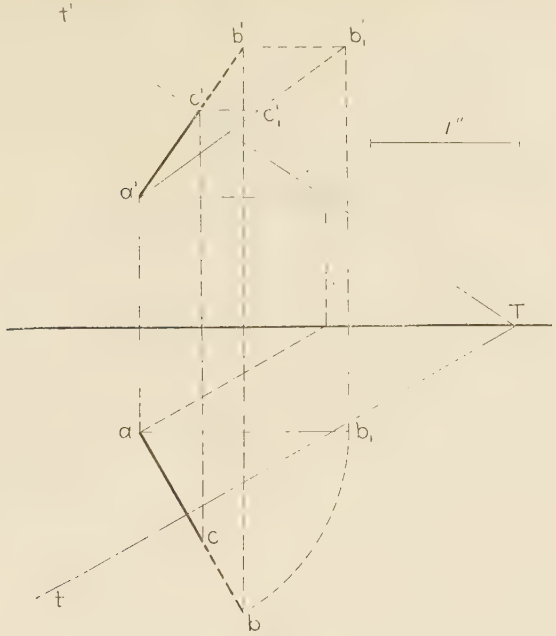


FIG. 38.

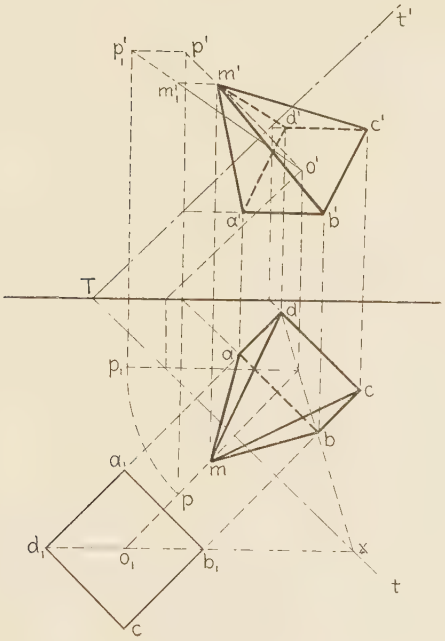


FIG. 39.

3. Connect the ends of these perpendiculars in order, thus drawing the opposite base.

If it be a pyramid,

2. Erect a perpendicular to the plane of the base of the specified altitude at the center of the base.

3. Connect the end of the perpendicular, the apex of the pyramid, with the corners of the base.

Note.—In the case of curved-surfaced solids, the procedure is the same, but only the outside elements, those tangent to the curves, are drawn.

Construction.—Let it be required to draw a square pyramid, of 1-in. base and 2-in. altitude, with the center of the base at O in the plane T, Fig. 39.

1. Revolve O to o_1 about the *H*-trace as an axis.

Note.—Under certain conditions it will be preferable to rotate about the *V*-trace, and under others it will be preferable to rotate about the *P*-trace. The student will have to judge from the layout of his problem.

2. Construct the square in its true size, $a_1b_1c_1d_1$, about o_1 as a center.

3. Counter-revolve the square into T (Problem 19).

4. Erect the perpendicular OP at O.

5. Revolve OP parallel to *V* (or *H*).

6. Lay off 2 in. on $o'p'_1$, at the point m_1.

7. Counter-revolve m_1 to m'_1m in $o'p'$, op.

8. Connect M with A, B, C and D, and make the invisible lines dotted.

75. EXERCISES

276. Through A(4+1½−1) draw a perpendicular to the plane T(1+2) 3(3−2), and find where it pierces it.

277. Through B(2+2−1½) draw a perpendicular to the plane S(4½+3) 2(4½−2), and find where it pierces it.

278. Through C(3+2−1) draw a perpendicular to the plane R(¾+1¼) 2(5−3), and find where it pierces it.

279. Through D(3+2−3) draw a perpendicular to the plane Q(+1½)∞ (−1), and find where it pierces it.

280. Through E(3+1½−2) draw a perpendicular to the plane X(1+3) 5(1+1½), and find where it pierces it.

281. A tower, which is a hexagonal pyramid of 16 ft. base, and 20 ft. altitude, has the center of its base at A(24 ft., 0−16 ft.). From a point O(38 ft. +11 ft. −8 ft.) a telephone wire is to be run into the tower perpendicularly to the nearest face. Locate the drilling point, and measure the length of the wire from O to the tower. Scale ⅛ in. = 1 ft.

282. From F(2, x−2) in the plane S(1+2) 3(3−2) erect a perpendicular 2 in. long.
283. From G(4+1, x) in T(4½+3) 2(4½−2) erect a perpendicular 1½ in. long.
284. From K(3+1, y) in W(¾+1¼) 2(5−3) erect a perpendicular 2 in. long.
285. From M(3+½, z) in X(+1½)∞(−2) erect a perpendicular 1½ in. long.
286. From N(2+1, x) in R(1+3) 5(1+1½) erect a perpendicular 2 in. long.

76. EXERCISES IN SOLIDS

In the following exercises only one or two solids are given to be erected on each type of plane. By interchanging solids and planes in different exercises, a large number of exercises may be very easily made up. **Consider the solids opaque, and dot all invisible lines.**

287. Erect an equilateral triangular prism, 1-in. sides and 2-in. altitude, on the plane T(1+2) 3(3−2), with the center of the base at A(2, x−2) in T.
288. Erect a square pyramid, 1¼-in. base and 2-in. altitude, on the plane T(1+2) 3(3−3). Center of base at B(2, y−1½).
289. Erect a 1½-in. cube on S(4½+3) 2(4½−2), with one corner at C(4+½, y) and another corner touching *H*.
290. Erect a rectangular pyramid, base 1 in. × 1½ in., altitude 2 in., on T(4½+3) 2(4½−2), one corner of the base at D(4+1, x), and the long edges of the base parallel to *H*.
291. Erect a pentagonal pyramid, base radius ¾ in., altitude 2 in., on R(¾+1¼) 2(5−2), center of base at E(3+1, y).
292. Erect a hexagonal prism, base 1½-in. long diameter, altitude 1½ in. on R(¾+1¼) 2(5−3), one edge of the base parallel to *V*, and ½ in. from *V*.
293. Erect a hexagonal pyramid, 1¼-in. short diameter of base, 2½-in. altitude, on P(4+3) 4(4−3), with the center of the base at M(4−1+1¼), and one edge parallel to *H*.
294. Erect a 1¼-in. cube on the plane Q(+1½)∞(−1½), with the center of the base at N(3+1, x), and the diagonal parallel to *H*.
295. Erect a cone, 1½-in. base and 2-in. altitude, on the plane Q(+1½)∞ (−1½), with the center of the base at N(3+1, x).
296. Erect a cylinder, 1-in. base and 2-in. altitude, on the plane X(1+3) 5(1+1½), with the center of the base at O(2+1½, x) in the plane X.
297. The points A(1¼+1−⅝), B(1¾+¼−¾) and C(2¼+¾−¼) limit the base of a triangular prism of 2-in. altitude. Draw its projections.
298. The line D(2+2−1) E(1½+1−½) is one edge of the base of a rectangular prism. The opposite edge runs through the point F(2+1−½). Its altitude is 2 in. Find its projections.
299. The lines A(3+1−1) B(2+¾−½) and BC(3+0−¼) are two sides of a parallelogram, which is the base of a prism 2 in. long. Draw the projections of the prism.
300. The apex of a pyramid is A(2+2−1), and its base is a 1 in.×1½ in. rectangle in the plane T(4+3) 2(4−2). Draw its projections.
301. The apex of a cone of 1½-in. base diameter is M(3+2−2). Its base is in S(1+2) 3(3−3). Draw its projections.

302. $O(2 \cdot 1 - 1)$ $P(3\frac{1}{2} \cdot 1 - 1)$ is the edge of a cube. One edge of the cube is in H. Draw its projections.

303. The apex of a $1\frac{1}{2}$-in. square pyramid is $A(3 \cdot 2 - 2)$, and its base is in $R_1 \cdot 2)(\infty - 1)$. Draw its projections.

304. Draw a line from $A(3\frac{1}{2} \cdot \cdot \cdot)$ and shall [illegible] $N(2 \cdot 1\frac{1}{2} \cdot 1)$ in a point P, and be $1\frac{1}{2}$ in. distant from the point $B(2 \cdot 2 \cdot 1\frac{1}{2})$.

77. **Problem 27. To pass a plane through a given point perpendicular to a given line.**

This problem is similar to Problem 24, in that, the required plane being perpendicular to the given line, its traces must be respectively perpendicular to the projections of the given line. Therefore, we know the **direction** of its traces.

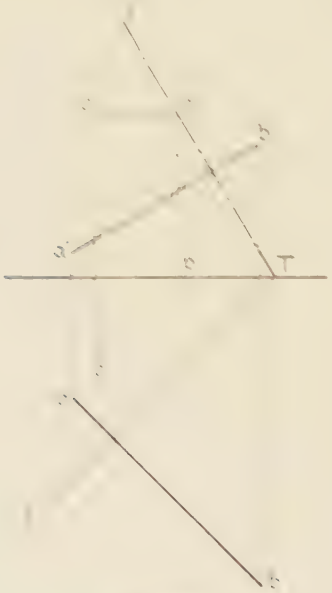


FIG. 40

Analysis.—1. Draw a line through the given point perpendicular to the given line and parallel to H, or it could be V. *See Theorem XX, Art. 24.

2. Find the V-piercing point of this auxiliary line.

3. Pass a plane through the auxiliary line, whose traces are respectively perpendicular to the projections of the line.

Proof.—It can be proved by the answers to these two questions: Does this plane contain the given point? Is this plane perpendicular to the given line?

Construction.—Let it be required to pass a plane through O (Fig. 40) perpendicular to AB.

1. Draw op perpendicular to ab.
2. Draw o′p′ parallel to GL.
3. Find P, the V-piercing point of OP.
4. Through p′ draw t′T perpendicular to a′b′.
5. Through T draw Tt perpendicular to ab. The plane T will contain the point O, and will be perpendicular to AB.

Notes.—1. If the given line lies in a profile plane, the problem can only be solved by the P-projection.

2. Under certain conditions (that is, where T falls outside the limits of the problem) it will be necessary to draw two lines like OP, one parallel to V, and the other parallel to H.

78. Problem 28. To find the distance from a point to a plane.

Analysis.—1. Erect a perpendicular from the point to the plane.

2. Find the piercing point of the perpendicular with the plane.

3. Measure the distance from the given point to the piercing point. This will be the distance.

Let the student make the construction.

79. Problem 29. To project a line on any oblique plane.

The projection of any point on any plane is the point in which that plane is pierced by the perpendicular from the given point.

Analysis.—1. Let fall perpendiculars from any two points (preferably the limiting points) of the line to the plane.

2. Find the piercing points of the perpendiculars with the given plane.

3. Join these piercing points with a line. This line will be the required projection.

Let the student make the construction.

Check.—It is obvious that the projection of a point lying in a plane is the point itself. It is also obvious that the point, in which a line pierces a plane, is identical with the projection of that point. Therefore the line will intersect its projection, on any plane at its piercing point with that plane. Hence,

it is probable that the projection is correctly made, if the line and its projection intersect at the piercing point.

80. Problem 30.—To measure the angle between any line and any plane.

First Method.—The measure of the angle between a line and a plane is the angle between the line and its projection on that plane. Therefore one analysis suggests itself; as follows,

Analysis.—1. Project the line on the plane by Problem 29.

2. Measure the angle between the given line and its projection thus found, by Problem 14.

Let the student make the construction.

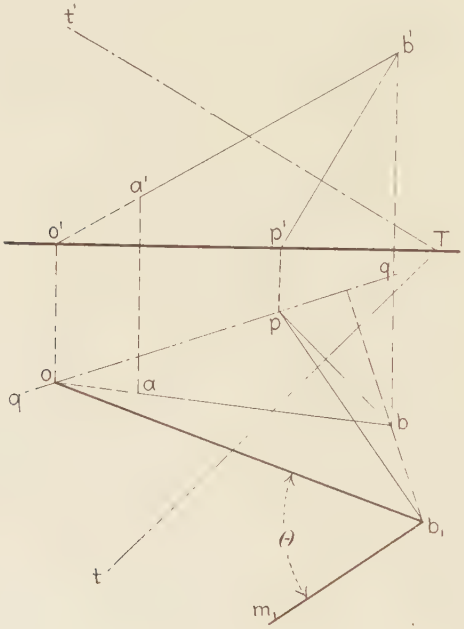


Fig. 41.

81. Second Method.—This method, while not so obvious is much quicker and better.

Analysis.—1. From any point in the given line erect a perpendicular to the given plane.

2. Measure the angle between the given line and this perpendicular.

3. Subtract this angle from 90°, which gives the required angle between the line and plane.

Proof.—By letting fall a perpendicular to the plane, a right-angled triangle is formed between the given line, its projection, and the perpendicular. The angles between the line and its projection, and between the line and the perpendicular are complementary, and add up to 90°. Hence this method might be called the "complementary method."

Construction.—Let it be required to measure the angle between AB (Fig. 41) and the plane T.

1. From B let fall a perpendicular, b′p′, bp, to T.

2. Find the *H*-piercing points (or *V*-piercing points, if more convenient) of the lines AB and BP.

3. Revolve B into *H* about the *H*-trace of the plane Q thus found.

4. Lay off a perpendicular, b_1m_1, to pb_1, and θ will be the required angle.

Compare this method of obtaining the angle with the "natural" method on the same exercise, and a considerable saving will be shown.

82. EXERCISES

305. Draw the traces of a plane T, perpendicular to A(2+½−½) B(4+2−1), through the point O(1+1½−2).

306. Draw the traces of a plane S, perpendicular to C(2+3−1) D(4+1−3), through the point P(3+1−3).

307. Draw the traces of a plane X, perpendicular to E(2+1−1) F(4+1−2), through the point M(3+1−3).

308. Draw the traces of a plane W, perpendicular to G(3+½−1) K(3+2−2), through the point P(3+3−1).

309. Draw the traces of a plane Y, perpendicular to A(2+1−1) B(4+1½−2) at its middle point.

310. Draw the traces of a plane Q, perpendicular to C(2+1−3) D(4+2−1), through the point D.

311. E(2+3−3) F(4+1−2) is the base of an isosceles triangle, whose vertex is in *GL*. Draw its projections and revolve to show its true size.

312. G(2½+1−2) K(3½+½−1) is one edge of a cube, one of whose corners rests on *H*. Draw the projections of the cube. Dot all invisible lines.

313. The center line of a stick 1½ in. square is the line A(2+1−1) B(4+3−3). Two ½-in. holes are drilled straight through the stick, ¾ in. from the respective ends. Draw the projections of the stick and holes, dotting all invisible lines.

314. Draw the projections of a hexagonal prism, 1½-in. long diameter, 1½-in. altitude, whose center line is C(2+3−1) D(4+1−3).

315. E(3+2−½) F(3+½−2½) is the axis of a 2-in. cylinder. Draw the projections of the cylinder.

316. G(2+½−1) K(4+2−2) is the axis of a right circular cone of 2-in. base. Draw its projections.
317. The upper surface of a vein of coal is found to be perpendicular to a shaft bored in the direction of A(2½+1−1) B(4+3−2). The surface is located at C(3, x, y) on AB. Find the traces of the plane of the ore.
318. A guy-wire supporting a stack runs from D(1½+3−1) on the stack to E(4+1−2) on a roof perpendicular to the wire. Find the traces of the plane of the roof.
319. A shaft center line is G(2½+1−3) K(2½+3−1½). A 12-in. pulley, with 6-in. face, is located in the center of GK. Draw the projections of the pulley (solid, without spokes), to a scale of 2 in. = 1 ft.
320. Find the distance from A(4½+2−1) to the plane T(1+3) 4½(4½−3).
321. Find the distance from B(4+2−1) to the plane S(1+3) 4(1−3).
322. Find the distance from C(4+2−2) to the plane R(2+3) 3(2−2½).
323. Find the distance from D(3+3−3) to the plane Q(+2)∞(−1).
324. Find the distance from E(5+1−1) to the plane W(1+2) 3(5−2).
325. Find the distance from F(3½+1½−¾) to the plane X(2+3) 5(2−½).
326. Find the distance from G(3+1−1) to the plane Y(1+3)(5+2)(1−3)(5−2).
327. A 3-in. cube is placed on H, with one face against V. Find the distance from the upper rear right-hand corner to the plane of the three adjacent corners.
328. An inclined roof is in the plane T(1+2) 5(1−3). From a point M(2, 0--1) in the floor it desired to run a cylindrical flue 8-in. diameter through the roof, at right angles to it. What is the length of the flue? Draw the projections of the opening in the roof. Scale 1½ in. = 1 ft.
329. Project M(2+2−1) N(4½+2−1) on the plane T(5+3) 1(5−3), and find the true length of the projection.
330. Project A(2¾+½−1½) B(3½+1½−½) on S(4+1½) 3(2−1½), and find where AB pierces S.
331. Project C(3+1−½) D(4+2−2) on R(+1)∞(−1½).
332. Project C(3+1−½) D(4+2−2) on P(4½+3) 4½(4½−3).
333. Project E(3+2−2) F(3+½−1) on Q(+1½)∞(−2).
334. Project the triangle G(2+1−2) H(3+2−1) K(3½+1½−1½) on the plane S (+½)∞(−2½).
335. Measure the angle that A (1+1½−1) B(3+1½−1) makes with Q(1+3) 5(1−2).
336. Measure the angle that C(1+2−2) D(3+1½−½) makes with R(4+3) 1(4 1).
337. Measure the angle that E(1+2−2) F(4+1−1) makes with S(3+3) 3(5−2).
338. Measure the angle that G(1+3−2) K(3+1−1) makes with the profile plane at P(3, 0, 0).
339. Measure the angle that M(2+2−3) N(4+1−½) makes with T(+1½) ∞(−1).
340. Measure the angle O(2+3−2) P(3½+1−½) makes with W(3½+3) 3(3½ 3).

341. Measure the angle that A$(2+1-\frac{1}{2})$ B$(4+1-\frac{1}{2})$ makes with X$(1+1\frac{1}{2})$ $3\frac{1}{2}(5-1\frac{1}{2})$.
342. Measure the angle that C$(3+1-1)$ D$(4+2-3)$ makes with Y$(1+3)$ $(5+2)(1-3)(5-2)$.
343. Measure the angles that E$(2+2-2)$ F$(3\frac{1}{2}+\frac{1}{2}-1)$ makes with *H*, *V* and *P*.
344. Measure the angle that the center line of the flue in Exercise 328 makes with the floor.
345. Find the angles that the guy-wire in Exercise 318 makes with *H* and *V*.
346. In Exercise 316, what angle does any element of the cone make with the plane of the base?
347. What angles does the telephone wire in Exercise 281 make with *H*, *V* and *P*?
348. What angles does the hip rafter EG in Fig. 25 make with the roof C, and with the floor?

83. Problem 31. To draw a line through a point making a given angle with a given plane, and intersecting a given line in the plane.

This problem is a special case of Problem 30, and would have an infinite number of solutions, but for the last condition introduced; that is, intersecting some line in the plane. This limits the solution to two possible lines, or even one, and, if the line be too far away, the problem is impossible.

Analysis.—The locus of all the lines making the given angle with the given plane would be the surface of a right circular cone, its apex at the given point, and its base in the given plane; hence,

1. Let fall a perpendicular from the given point to the given plane, and find its piercing point.

2. This point is the center of the base of an imaginary cone; therefore obtain the true length of the perpendicular.

3. With the revolved perpendicular as one side, construct a right-angled triangle with the given angle at the base.

4. With the base of this triangle as a radius, draw an arc from the piercing point of the perpendicular as a center, intersecting the given line.

5. Connect the given point with the intersection of the arc and the given line. This line will fulfil the conditions of the problem.

Construction.—Let it be required to draw a line from O (Fig. 42), making 60° with T, and intersecting the line AB, that lies in T.

1. Draw OP, perpendicular to T, from O.
2. Find the piercing point, P, of OP with T.
3. Measure OP (o'_1p').
4. Construct a right triangle $p'mo'_1$, with the angle at $m = 60°$.
5. Revolve P and AB into *H* about Tt.
6. From p_1 as a center and with o'_1m as a radius, strike an arc, intersecting ba_1 at c_1.
7. Counter-revolve c_1 to cc′ on AB.
8. Join O with C. OC is the required line.

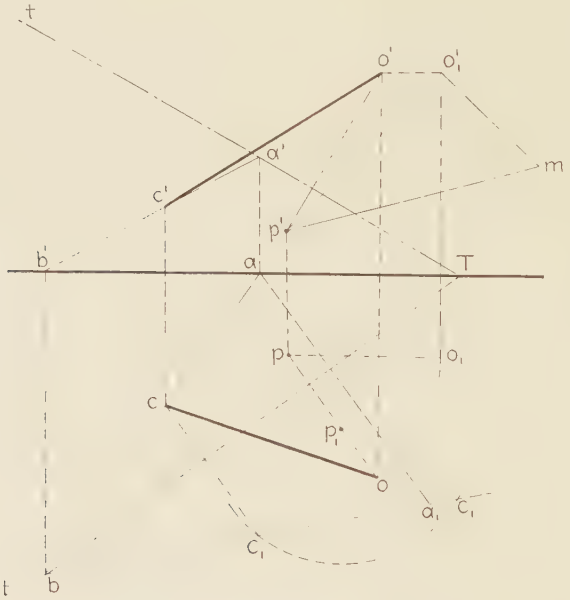


FIG. 42.

84. Problem 32.—Draw a line through a given point that shall make given angles with H and V.

Note.—The sum of the two given angles may be any amount between 0° and 90°. Any line parallel to *GL* makes angles with *H* and *V* equal to zero, and any line in a profile plane makes angles with *H* and *V* that add up to 90°.

Analysis.—The analysis of this problem, and several problems later, depends on certain properties of the right circular cone, so it is best to discuss these before undertaking the construction of the problem.

DISCUSSION OF THE RIGHT CIRCULAR CONE.

Three properties of the cone are particularly useful in the solution of certain problems.

1. The elements of the cone all make the same angle with the plane of the base, and they are all of the same length, because one way of generating a cone is to revolve a right-angled triangle about one of its perpendicular sides.

2. All planes tangent to a cone make **the same angle** with the plane of the base, because a tangent plane is perpendicular to the generating triangle of the cone, and contains its hypothenuse.

3. Every element of the cone and every plane tangent to the cone pass through the apex.

85. Construction of Problem 32. – Let it be required to draw a line through A, Fig. 43, that shall make 45° with H, and 30° with V.

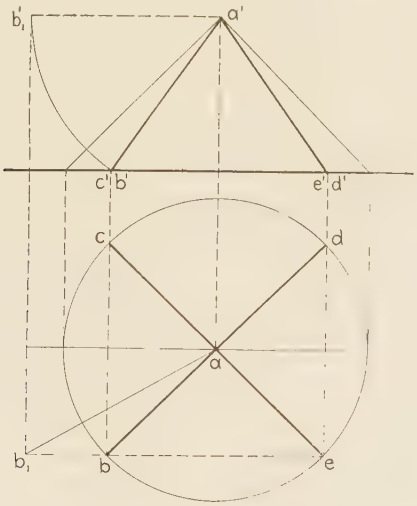


Fig. 43.

1. Make A the apex of a right circular cone, whose base is in H, and whose elements make 45° with H.

All the elements make 45° with H, and all pass through A. Therefore, it remains to be found which are the four possible elements that make 30° with V.

2. From A draw a line AB parallel to H, making 30° with V, and equal in length to the elements of the cone, its projections will be ab_1, $a'b'_1$.

3. Revolve B about an axis through A perpendicular to V, until it comes into contact with the cone at ab_1 $a'b'$. That is, when the circular path of B cuts the circular base of the cone. AB is the required line.

Notes.—1. ABB_1 may be regarded as part of the surface of a second cone, and AB will then be the line of intersection of the two cones, the common element.

2. **Four lines** may be drawn through A, all making 45° with H, and 30° with V, AB, AC, AD and AE, all shown in Fig. 43.

86. EXERCISES

349. Draw a line from A(4+2−1) making 60° with T(1+3) 5(1−2), and intersecting B(2, 0, x) C(3, y, 0) in T. What angle does the line make with BC?

350. Draw a line from D(3½+2−2) making 30° with S(1+3) 3(5−1), intersecting E(2, x, 0) F(4½, 0, y) in S. What angle does the line make with EF?

351. Draw a line from G(3+2−1) making 30° with R(+1½)∞(−1), intersecting H(2+1½, 0) K(4, 0−1). What angle does the line make with HK?

352. Through A(3+2−1) draw four lines, each making angles of 60° with V and 15° with H. What angles do they make with one another?

353. Through B(3+1½−1½) draw four lines, each making angles of 20° with V and 35° with H.

354. Draw a line 3½ in. long, terminating in H and V, making angles of 45° with V, and 30° with H.

355. Draw a line through C(4+1−1) making 50° with H and 40° with V.

356. Draw a line through D(3, 0, 0) making angles of 45° with H, and 30° with V.

357. Through E(1+1½−1) F(4+1½−1) pass a plane perpendicular to T(1+3) 5(1−2).

358. Through G(2+3−3) H(4+1−½) pass a plane perpendicular to S(+2)∞(−1).

359. From a point K(16 ft. +24 ft. −16 ft.) on a stack, a guy-wire is run to a point on a horizontal roof 8 ft. above H. The guy-wire inclines 60° to H and 20° to V, running to the southeast. Where does it strike the roof, and what is its length? Scale ½ in. =1 ft.

87. Problem 33. To pass a plane parallel to a given plane through a given point.

Two parallel planes must have their respective traces parallel. The converse is not always true. For example, witness any two planes, whose traces are parallel to GL. They may, or may not, be parallel.

Analysis.—1. Draw a line through the given point parallel to the H-trace of the given plane.

2. Through the V-piercing point of this auxiliary line draw the V-trace of the plane parallel to the V-trace of the given plane.

3. From the intersection of the V-trace and GL draw the H-trace parallel to the H-trace of the given plane.

Construction.—Let the student make the construction and draw the figure. Note the similarity in this problem and in Problem 27.

Note.—If the GL intersection of the traces is not accessible, two auxiliary lines may have to be drawn.

Special Case.—When the given plane is parallel to GL. In this case the profile view must be used. The P-traces of the given and required planes must be parallel.

88. Problem 34.—To pass a plane parallel to a given plane at a given distance from it.

Analysis.—1. From any point in the given plane erect a perpendicular of a length equal to the given distance according to Problem 25.

2. At the other end of this perpendicular draw a plane parallel to the given plane, according to Problem 33.

The similarity between this problem and the preceding one is so obvious that no construction or figure is necessary.

Second Method. Analysis.—1. Pass an auxiliary plane perpendicular to the H-trace of the given plane.

2. Revolve the intersection of the two planes about the H-trace of the auxiliary plane.

3. Draw a parallel to the revolved intersection at the required distance.

4. Counter-revolve this line and draw the required traces through its piercing points, parallel to the given traces.

This method will be found easy and convenient. Let the student make the construction.

89. EXERCISES

360. Through A(4+1−1) pass a plane parallel to T(1+3) 3(1−1). What is the distance between the planes?

361. Through B(2+1−2) pass a plane parallel to S(1+1) $5\frac{1}{2}$(1+3). What is the distance between the planes?

362. Through C(5+1−2) pass a plane parallel to R(2+3) 2(4−3). What is the distance between the planes?

363. Through D(5+1−1) pass a plane parallel to Q(4+3) 3(2−3). What is the distance between the planes?

364. Through E(4+1½−½) pass a plane parallel to X(1+3) 4(1−2). What is the distance between the planes?
365. Through F(5+1−2) pass a plane parallel to W(+2)∞(−1). What is the distance between the planes?
366. Draw the traces of a plane 1 in. from Q(1+3) 3(1−1) and parallel to it.
367. Draw the traces of a plane 1½ in. from R(2+3) 2(4−2) and parallel to it.
368. Draw the traces of a plane 1 in. from S(1+1) 5½(1+3) and parallel to it.
369. Draw the traces of a plane 1¼ in. from T(4+2) 3(2−2) and parallel to it.
370. Draw the traces of a plane 1 in. from W(1+1) 5(1−3) and parallel to it.
371. A saw-toothed roof is being erected for a greenhouse. The plane of one roof plane is T(1+2) 3(1−2½). Find the traces of two other roof planes, the distance between any two adjacent parallel planes being 7 ft. Draw to a scale of 1½ in. = 1 ft.
372. The alternate planes in the roof in Exercise 371 are perpendicular to those already found. The ridge line is a horizontal line, 8 ft. above *H*. Construct the *H* and *V*-projections of as much of the roof as possible in the space. Scale ½ in. = 1 ft.

90. Problem 35. – To find the true angle between two planes. The angle between two planes is considered to be equal to that angle included between the perpendiculars to the line of intersection, that lie in each plane. It is evident, then, that the plane of the measuring angle is perpendicular to the line of intersection, and therefore to both of the given planes. Two methods of finding this angle are given herewith.

First Method. Analysis.—1. Find the intersection of the given planes.

2. At any assumed point on the line of intersection pass a plane perpendicular to it.

Note.—Only one trace of this auxiliary plane is usually necessary.

3. Measure the angle between the intersections of the auxiliary plane with the two given planes.

Construction.—Let S and T (Fig. 44), be the two planes.

1. Obtain their line of intersection, AB.

2. Through O, assumed on AB, draw OP, perpendicular to AB and parallel to *V*.

3. Obtain P, the *H*-piercing point of OP.

4. Draw qq through p, perpendicular to ab.

5. Join o with x and y (where qq crosses Tt and Ss, respectively). OX and OY are the perpendiculars to AB, lying in T and S.

6. Revolve O about qq into H, and connect o_1 with x and y. The angle θ (xo_1y) is the true measure of the angle between the planes.

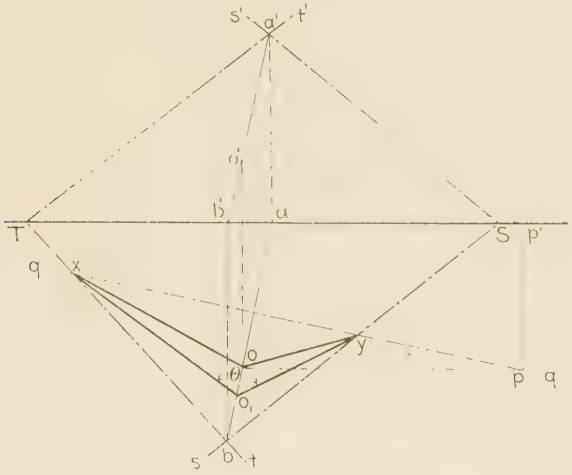


Fig. 44.

91. Second Method. Analysis.—1. From any point let fall two perpendiculars, one to each of the two given planes.

2. Measure the angle included between these perpendiculars. This will be the correct measure of the required dihedral angle.

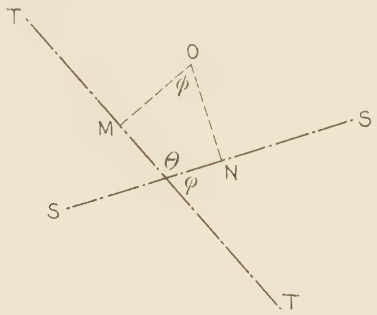


Fig. 45.

Proof.—The two perpendiculars from the point to the planes determine a plane perpendicular to both planes. Fig. 45 shows a view of two intersecting planes S and T, looking along their intersection. O is taken anywhere, and perpendiculars OM and ON are let fall to T and S respectively. The plane of MON cuts

lines MP and NP from T and S, and the four lines, OM, ON, MP and NP form a quadrilateral, the sum of whose angles is 360°. Prove this. The angles at M and N are right angles by construction, and therefore add up to 180°. This leaves 180° as the sum of angles ϕ and θ, at O and P respectively. The angle ϕ is therefore the supplement of θ, which is the angle between T and S. But the measure of any angle, plane or dihedral, may be either the acute angle or its supplement, so the angle between the planes S and T may be either θ and ϕ, and as ϕ is the angle between the perpendiculars, it is also the measure of the given dihedral angle.

Note.—Although the angle between two lines, or two planes, may be either the acute or the obtuse angle, as stated in the foregoing, the acute angle is usually taken as the true measure.

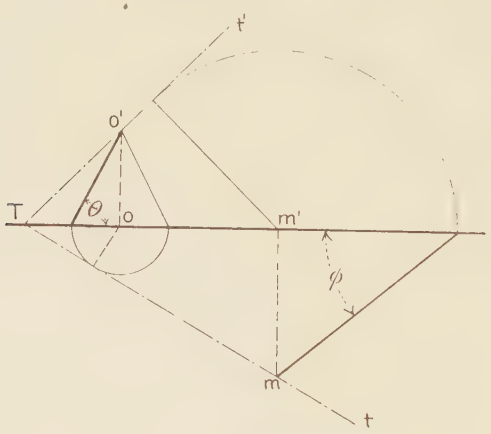


Fig. 46.

92. Problem 36.—To find the true angle between a plane and one of the projecting planes.

Analysis.—To find the angle between T and H.

1. Let any point in the V-trace of T be the apex of a right circular cone, whose base is in H.

2. Let the circular base of the cone be tangent to the H-trace of T.

3. The angle of the elements with H will be the angle between T and H.

Referring to Art. 84, the Discussion of the Cone, we shall find that any plane tangent to a cone makes the same angle with the plane of the base that is made by the elements with the base.

Fig. 46 shows such a cone with its apex at o′ in the V-trace, and its base tangent to the H-trace. The angle θ is therefore the true angle between T and H.

Fig. 46 also shows a cone with its apex at m in the H-trace of T, and its base tangent to the V-trace. Its angle, ϕ, is the true angle between T and V.

93. Problem 37.—Given one trace of a plane and its angle with the plane of projection, to find the other trace.

Analysis.—1. Construct a cone, whose elements make the given angle with its base, tangent to the given trace, with the center of its base in GL.

2. Draw the other trace through the apex of the cone and the intersection of the given trace with GL.

Notes.—1. If the given trace does not intersect GL within bounds, two cones can be constructed, and the trace drawn through the two vertices.

2. If the given trace is parallel to GL, the other trace will also be parallel, and the quickest solution is effected through the profile plane.

3. By inverting the cone, another plane could be found that would fulfil the conditions.

94. Problem 38. The H-trace of a plane being given, and its angle with V, to find the V-trace.

Analysis.—1. From any point in the given H-trace draw a line making the given angle with GL.

2. From the same point in the H-trace, draw a line perpendicular to GL.

3. Draw a semi-circle in V, having the second point on GL as its center, and the first point on GL as the other end of the radius.

4. Draw the V-trace tangent to the circle from the intersection of the H-trace and GL.

Note.—This solution is the same cone construction as employed in previous problems.

Construction.—Let Tt, Fig. 46, be the given trace, and let ϕ be the angle the plane makes with V.

1. Assume M, any point on Tt.

2. Lay off mn, making the angle ϕ with GL.

3. With m′ as a center and m′n′ as a radius, draw the semi-circle, the base of the cone.

4. From T draw a tangent to the semi-circle.

This will be the required V-trace.

Note.—In this problem, as in Problem 37, two solutions are always possible, except when the angle given is a right angle. This is accomplished by inverting the cone in Problem 37; that is, using the same base, but drawing the apex below *H;* and in Problem 38 it is effected by completing the circle, and drawing the tangent on the opposite side.

95. EXERCISES

Find the true dihedral angle between the planes given in the following exercises.

373. T(4+3) 2(5−2) and S(1+3) 5(1−2½).
374. Q(1+2½) 4½(1−2) and R(4+3) 2(2−3).
375. W(+1½)∞(−1) and X(5+3) 2(5−1½).
376. Y(+1½)∞(−1) and Z(1+2½) 3(5−2½).
377. Q(2+3) 4(5−1½) and R(5+3) 2¾(½−3).
378. S(+2)∞(−½) and T(+¾)∞(−1¾).
379. W(3+3) 3(3−3) and X(1+2) 5(1+1).
380. Y(1+3) 3(1−2) and Z(5+2) 3(5−1½).
381. Q(1+3) 2½(1−1) and R(4+3) 3(5−2).
382. S(+2)∞(−1) and P(3+3) 3(3−3), *H* and *V*.
383. T(4+3) 2(5−2) and *H*, *V* and *P*.
384. Q(+1½)∞(−2) and *H* and *V*.
385. S(1+2) 5(1+1) and *H*, *V* and *P*.
386. W(1+2) 4(1−3) and X(5+3) 4(1−3).
387. Given A(2, 0−1½) B(3+1, 0) and a plane S(3+1) 5(2−1½). Pass a plane T through AB that will make 60° with S.
388. Pass a plane Q through C(2+2−½) D(3+½−1½) which will make 45° with the plane R, which contains CD and is parallel to *GL*. Find the traces of Q.
389. Find the traces of two planes containing the line C(2+2−½) D(3+½−1½), each making 60° with *H*. Measure the angle between these planes.
390. Through the line E(3+½−2) D(4½+1¾−½) pass a plane S making 60° with *H*, and a plane T that makes 60° with S.
391. Measure all the dihedral angles of the tetrahedron A(2, 0−1) B(4, 0−1) C(4, 0−2¼) D(2½+1½−2).
392. Measure the hip roof angle between the roofs C and D in Fig. 25, Article 54.
393. A square chute has its center line A(4+1½−½) B(2, 0−2). The diagonal of its right section is parallel to *H*, and its sides are each 12 in. in width. What angle does any one of its sides make with the floor? Draw the intersection of the chute with *H*. Draw the chute to a scale of 1½ in. =1 ft.
394. A tower is designed in the form of a hexagonal pyramid, having a 16 ft. base and an altitude of 20 ft. The framework is of structural iron. At what angles must the angle irons for base and rafters be rolled?
395. Draw the traces of two planes, S and T, making angles of 45° with each other, containing the line M(2, 0−2) N(3+1½, 0).

Note.—Projecting planes are not intended to be used as either S or T in this exercise.

396. Find the traces of two planes making 60° with each other each containing the line O(2+1−1) P(4+1−1), one of them being parallel to A(1+2 −1) B(3+½−1½).
397. A plane makes 30° with H, and its H-trace is T(4, 0, 0) t′(1, 0−2). Find its V-trace.
398. Two planes, Q and R, have their H-traces in the line C(4, 0, 0) D(2, 0 −2). Q makes 45° with H, and R makes 75° with H. Draw their V-traces and measure the angle between Q and R.
399. Draw the traces of the bisecting plane between Q and R in Exercise 398.
400. The H-trace of a plane T is A(2, 0−½) B(4, 0−½) and the plane makes 15° with V. Find the V-trace. Another such plane could be found. Draw its V-trace.
401. The H-trace of two planes, S and R, is C(5, 0, 0) D(2, 0−2). Both planes make 60° with H. Draw their V-traces, measure the angle between them, and draw the V-trace of the bisecting plane of the angle.
402. The V-trace of two planes, X and Y is E(4, 0, 0) F(2+1½, 0). Both planes make 45° with H. Draw their H-traces.
403. The V-trace of two planes, Q and R, is the line G(2+3, 0) K(4+2, 0). Both planes make 30° with V. Draw their H-traces. Measure the angle between them.
404. A horizontal line in a vein of ore, that inclines 15° to H, runs from A(4, 0, 0) south, 42° west. Find the V-trace of the plane of the ore.
405. A(2, 0−2) B(5, 0, 0) is the line of the lower edge of a roof inclined 30° from the horizontal. The length of the roof is 30 ft., with a corner at A, and the ridge is 10 ft. above the lower edge. The roof is rectangular. Draw its H- and V-projections to a convenient scale.

96. Problem 39.—To find the traces of a plane passed through a given line in a given plane, so as to make a given angle with the given plane.

Analysis.—Assume a point in the given line, and pass a plane through the point perpendicular to the line.

2. Revolve the line cut by this auxiliary plane from the given plane about the trace of the auxiliary plane into H or V, as the case may require.

3. From the revolved position of the assumed point on the given line lay off a line, making the given angle with the revolved line.

4. Where this new line crosses the trace of the auxiliary plane will be a point on the trace of the required plane.

5. Since the required plane must contain the given line, its traces will pass through the H- and V-piercing points of the

given line. As we have these two points, and also the point on the required trace determined in operation (4), we have the three points necessary to determine the traces of the required plane.

Construction.—Refer to Fig. 44, Article 90. Let S be the given plane, AB the given line in S, and θ the given angle.

1. Find the piercing points of AB.
2. Assume O on AB.
3. Draw the *H*-trace of Q, containing O, perpendicular to AB.
4. Revolve OY about qq to o_1y in *H*.
5. Lay off from o_1 a line o_1x, which makes θ with o_1y.
6. Draw the *H*-trace of T through b and x.
7. Draw the *V*-trace of T through T and a′.

Notes.—1. Two planes, such as T, making θ with S can always be drawn, except when θ is 90°.

2. When it is required to pass such a plane perpendicular to the given plane, it is only necessary to draw a line perpendicular to the given plane at any point of the given line. The given line and its perpendicular intersector then determine the required plane.

97. Problem 40. —To find the traces of a plane passed through a given line, so as to make any required angle with H or V.

Limitation.—The given angle must be as large or larger than the angle that the given line makes with the plane.

Analysis.—1. Take any point on the line for the apex of a cone, having its base in the plane to which the angle is given, its elements making the given angle with that plane. See Article 84.

2. Find the piercing points of the given line and draw the traces of the plane which contains the line and is tangent to the cone.

Let the student make the construction.

98. Problem 41. To find the traces of a plane which shall make any required angles with H and V.

Limitations.—The sum of these angles must be equal to or more than 90°, or equal to or less than 180°. A plane parallel to *H* is 90° from *V* and 0° from *H*. A profile plane is 90° from *V* and 90° from *H*. Oblique planes will be between these limits.

Construction.—Let it be required to draw a plane making 60° with *H* and 45° with *V*.

1. Draw the projections of a sphere, having any radius, and its center at any point in *GL*. (See Fig. 47.)

2. Draw a 60° cone tangent to the sphere with its base in H and its apex in V at a′.

3. Draw a 45° cone tangent to the sphere with its base in V and its apex in H at b.

4. Draw the V-trace of the required plane through a′ tangent to the circle in V, and the H-trace through b tangent to the circle in H.

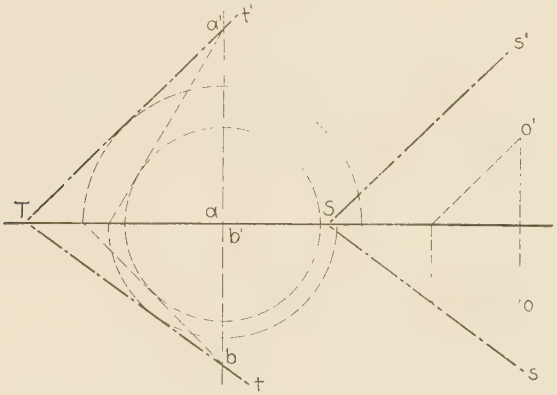


Fig. 47.

Notes.—1. Four planes fulfilling these conditions can be drawn, all tangent to the sphere.

2. Any plane parallel to GL has the sum of its angles with H and V equal to 90°, hence, if the sum of the given angles is equal to 90°, the problem can be easiest solved by the use of the profile plane.

3. If it be required to pass such a plane through a given point, an auxiliary plane may be drawn anywhere, as in the foregoing, filling the conditions, and then a plane may be passed through the point parallel to the auxiliary plane. The plane S, Fig. 47, is passed through the point O, parallel to T already found. The plane S makes the same angles with H and V that T does.

Second Method. (From Church and Bartlett.)—"If a line is perpendicular to a plane, the angles between the line and the planes of projection will be **complements** of the angles between the plane and the planes of projection. Hence if we construct a line, making with H and V the complements of the required angles, a perpendicular plane through this point will make the required angles with H and V."

99. EXERCISES

406. Through A(2+1½, x) B(3+½, y) in R(1+3) 1(4−2) pass a plane Q making 45° with R.

407. Draw the traces of a plane S, making 60° with T(3+2) 5(1½−2), containing the line C(3+2, 0) D(1½, 0−2).

408. Draw the traces of a plane W, making 30° with X(2+2) 4(2−1), through D(3, 0−x) E(3+y, 0) in X.

409. Draw the traces of a plane Q, making 75° with R(5+2) 3(1−2), through the line G(2, 0−1)H(5+2, 0).

410. Draw the traces of two planes, S and R, each making 30° with T(5+2) 3(1−2), containing the line K(4+1, 0) L(4, 0+1).

411. Draw the traces of two planes, W and Q, each making 45° with X (+2)∞(−1½), containing the line M(2+2, 0) N(3½, 0−1½).

412. Draw the traces of a plane Y, making 60° with Q(+2)∞(−1½), containing the line O(3+2, 0) P(3, 0−1½).

413. Draw the traces of a plane Q, making 75° with R(2+2½) (3½+2) (2−2½) (3½−2), containing the line A(2, 0−2½) B(3½+2, 0). What angle does Q make with *H* and *V*?

414. Draw the traces of a plane S, making 30° with T(2+2½) (3½+2) (2−2½) (3½−2), containing the line C(3½+2, 0) D(3½, 0−2). What angles do S and T make with *H*?

415. Given W(3+2) 5(1½−2). Pass a plane X that will make 45° with W and cut it in a line parallel to *H*, and 1 in. above *H*.

416. The plane of a bin cover is Y(+1)∞(−1½) and a conveyor belt, 16 in. wide, touches the point E(3+3−3), and runs through a horizontal slot in the bin cover, 4 in. × 18 in., at an angle of 120° with Y. Draw the projections of the slot. Scale ¾ in. = 1 ft.

417. Through A(2+1−1) B(4+3−2) pass a plane that will make 45° with *H*. What angle does it make with *V*?

418. Through C(3+2−½) D(3+½−1½) pass two planes making 60° with *H*, and measure the angle between them.

419. Through E(2+1¼−¾) F(4+1¼−¾) pass two planes, making 30° with *V*. What is the angle between them?

420. Through G(2+½−2) H(4+2−½) pass two planes, making 75° with *H*. What angles do they make with *V*, and with one another?

421. A hip rafter runs from M(2, 0−2) to N(4+2−1). Two roofs, each making 75° with *H*, are to be erected, meeting on this rafter. If the design of each roof is an isosceles triangle, what will be their *H*- and *V*-projections? What is the angle between the roofs?

422. Draw the traces of a plane making 60° with *V* and 30° with *H*, containing the point M(3+1−1).

423. Draw the traces of a plane making 75° with *V* and 45° with *H*, containing the point A(4+1−2).

424. Through M(3+1−1) draw the traces of all the planes possible that make 60° with *V* and 45° with *H*.

425. Through N(3+2−1) draw the traces of a plane, making 30° with *V* and 75° with *H*, using the second method.

426. A line shaft makes 15° with *H* and 45° with *V*, and has a 12-in. pulley,

6-in. face, with its center at O($3\frac{1}{2}+1-2$). Draw the projections of the pulley. Scale $1\frac{1}{2}$ in. = 1 ft.

427. A man is reading a newspaper, 18 in. × 24 in., and his eye, at P($2+2-1$), is directed toward the center of the paper, 2 ft. distant, at an angle of 30° with H and 45° with V. Draw the projection of the newspaper to a scale of 1 in. = 1 ft.

100. Problem 42. To find the shortest distance between two lines not in the same plane.

Analysis.—1. Pass a plane through one of the lines parallel to the other.

2. Find the distance from any point in the second line to this plane.

This will be the shortest distance between the lines.

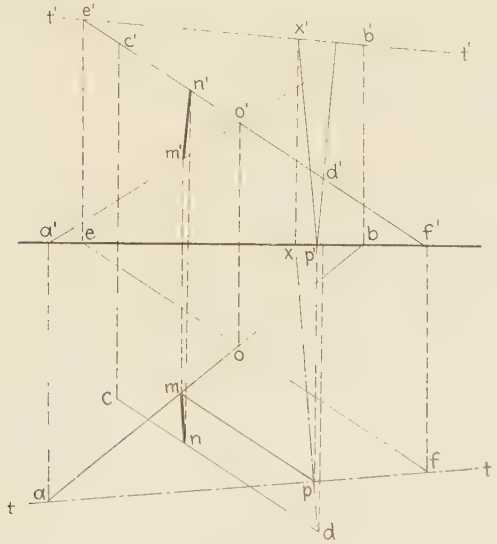


Fig. 48.

Proof.—The line being everywhere equally distant from the plane, there will be one point where a perpendicular from the line to the plane would intersect the line in the plane. This perpendicular would be the shortest distance between the lines, and would be equal to any perpendicular let fall from any point in the line to the plane.

Let the student make and explain the construction.

101. Problem 43. To find the shortest line that can be drawn between two lines not in the same plane.

This line is sometimes called the "common perpendicular" to the two lines.

Analysis.—1. Through one of the lines pass a plane parallel to the other line.

2. Project the second line on this plane. (Problem 29.)

3. Where the projection of the second line intersects the first line, erect a perpendicular to the plane, intersecting the second line.

This will be the required line.

Construction.—Let it be required to find the shortest line possible to be drawn between the lines AB and CD in Fig. 48.

1. Through any point, O, of AB, draw EF parallel to CD.

Note.—To avoid too many lines in the figure, the projection e'f' is drawn to coincide with c'd'. This is not absolutely necessary, however, but convenient.

2. Pass the plane T through AB and EF.

3. Project CD on T, in the line MP.

Note.—One projection only need be found.

4. At M, where MP intersects AB, erect a perpendicular, MN, to the plane T, intersecting CD at N.

The required line is MN. Let the student prove that MN will intersect CD.

102. EXERCISES

428. Find the shortest distance from A($1\frac{1}{2}+2-2$) B($3+\frac{1}{2}-1$) to C($5+1-1$) D($4+2-1\frac{1}{2}$).

429. Find the shortest distance from E($1+3-2$) F($2+1-2$) to G($3+1\frac{1}{2}-3$) H($2+1\frac{1}{2}-1$).

430. Find the shortest distance from K($2+1+1$) L($4+2+2$) to M($3-\frac{1}{2}-\frac{1}{2}$) N($4-2-2$).

431. Find the shortest distance from A($2+1-2$) B($2+3-2$) to C($3+2-1$) D($5+1-3$).

432. Find the shortest line parallel to *H* that can be drawn between E($1+\frac{1}{2}-3$) F($4+3-1$) and G($2+3-2$) H($3\frac{1}{2}+1-2$).

433. Find the common perpendicular to K($1+1-1\frac{1}{2}$) L($3+3-3$) and M($4+3-1$) N($5+1-3$).

434. Find the common perpendicular to A($1+2-3$) B($4+3-2$) and C($1+1\frac{1}{2}-\frac{1}{2}$) D($4+1-1\frac{1}{2}$).

435. Given a 3-in. cube on *H*. Find the shortest distance between the non-intersecting diagonals of the base and one of the adjacent faces.

436. Find the shortest line that can be drawn from E($2+1-3$) F($4+1-1$) to G($2+3-1$) H($4+2-1$).

437. Two wires run from K(1+3−3) to L(3+1−½) and from M(2+3−1) to N(5+1−2). Find where the shortest connecting wire can be placed, and its length. Scale 2 in. =1 ft.

438. A 2-in. pipe is run from a hole in a wall at O(3+3, 0) 45° to *H* and 45° to *V*. A 1½-in. pipe runs from P(1+3, 0) in the wall to M, 2 ft. straight out from the wall. It then is turned at right angles and runs downward to the right at 45° to *H*. Where can the shortest connection be made between the two pipes, and how long must this pipe be cut, allowing 2 in. at each end for connections? Scale ¼ in. =1 ft.

439. In a mill there is a bin, a 6-ft. cube, placed on the floor against the wall, with one edge at A(12 ft., 0, 0) B(18 ft., 0, 0). A 24-in. square chute has its center line C(3ft.+9 ft., 0) D(16 ft., 0−9 ft.) running from wall to floor. It is desired to run a 9-in. cylindrical pipe from a hole at E(18 ft. +9 ft., 0) to a hole in the floor at F(5 ft. 6 in. +0−14 ft.). Can the pipe pass both the bin and the chute? How much space will there be to spare in each case? Scale ¼=1 ft.

440. In a mill room there are two pipes with center lines respectively A(1+3−1) B(4, 0−1) and C(1, 0−1½) D(4+2−1½). Run a pipe from E(3+2, 0) in the wall to a point in the floor, so that its center line will be exactly 6 in. from AB and CD. Scale 1 in. =1 ft.

103. The Use of an Auxiliary Plane of Projection.

Many cases arise in practical work and in abstract problems where an auxlliary plane of projection may be used, often to considerable advantage. In the majority of cases this plane is taken perpendicular to *H* and oblique to *V*, yet many times it will be drawn perpendicular to *V* and oblique to *H*. It depends on the circumstances which decide the question. Both kinds will be used in our illustrations.

104. Illustration 1.—To find the line of intersection between a Hexagonal Prism in natural position and a Triangular Prism in oblique position.

Fig. 49 shows the layout, in **Third Angle** projection, of such a problem with the lines of intersection undetermined. The surfaces of the solids are entirely made up of planes, and, therefore, the lines of intersection will be made up of straight lines. By following the usual method of procedure, we could obtain the various lines of intersection, (1) by obtaining the intersections of the various planes, each with each, or (2) by obtaining the piercing points of the edges of one prism with the various faces of the other, and then connecting these piercing points.

In many cases the "usual" method is difficult or cumbersome, and the use of the auxiliary plane is a simple and quick method.

Construction.—The auxiliary plane to use in this case is one parallel to the bases of the oblique prism. In this case its bases are oblique to H and perpendicular to V. Fig. 50 shows the

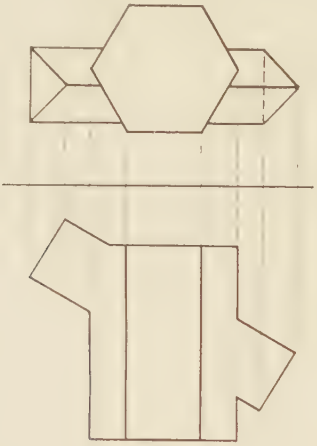


Fig. 49.

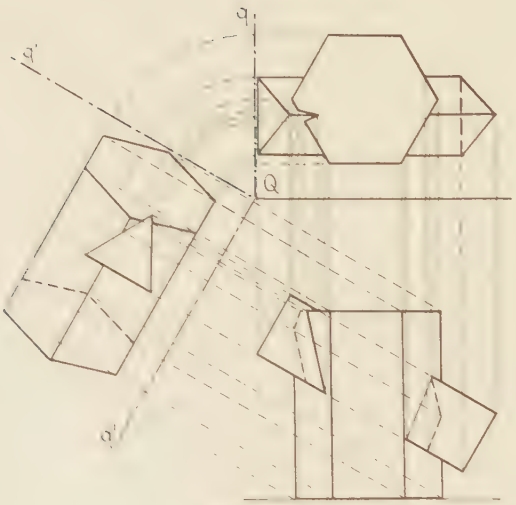


Fig. 50.

plane Q revolved into V about its V-trace. The H-trace revolves into a position perpendicular to the V-trace, to the position Qq_1.

The two solids are projected on Q by projecting the points perpendicularly to the respective traces of Q, in the same way

that a P-projection would be made, excepting that **the points are rotated through an acute angle,** instead of 90°.

The triangular base of the oblique prism appears in this projection in its true size, and this is the correct way of obtaining its H- and V-projections; that is, by drawing it first on Q, and projecting from that plane to H and V.

The intersecting points are easily located from the Q-projection and projected back to the surfaces in their natural position.

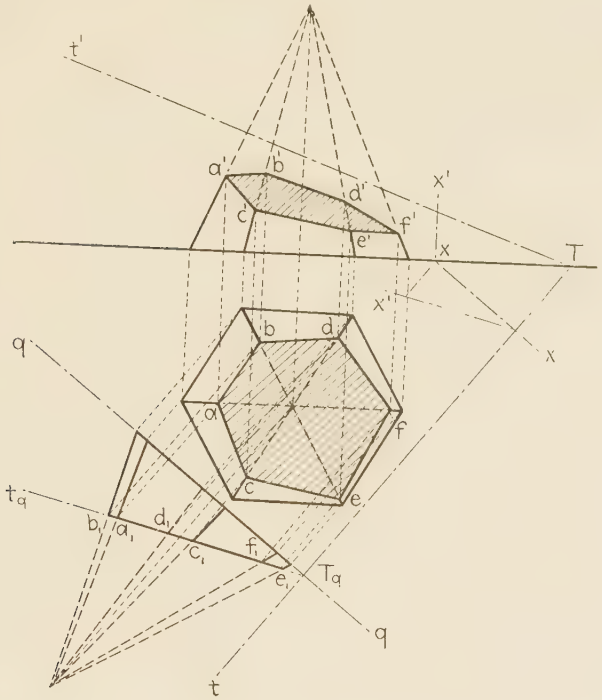


Fig. 51.

105. Illustration 2. To find the intersection of a plane and a pyramid.

Let the Hexagonal Pyramid and the Plane T, Fig. 51, be the given surfaces.

Discussion.—If a plane Q perpendicular to the H-trace of T be employed as an auxiliary plane of projection, it will also be perpendicular to H, and may be regarded as a vertical plane of projection with its H-trace qq as the ground line. The new

ground line may be placed at any distance, or on either side of the *H*-projection, wherever there is a good open space. The advantage of this arrangement is that T is a **projecting plane** on Q, and consequently all points in T will be projected on Q in its Q-trace.

Analysis.—1. Draw the *H*-trace of Q, perpendicular to the *H*-trace of T.

2. Project the pyramid on Q.

Note.—All points projected on Q are the same distance from qq that the *V*-projections are from *GL*.

3, Obtain the Q-trace of T.

Note.—If the intersection of the Q and T vertical traces is outside the limits of the problem, an auxiliary plane X may be passed parallel to Q, and the Q-trace of T will be parallel to the X-trace of T, see Xx'_1. The Q-trace of T can then be drawn.

4. The points a_1, b_1, c_1, etc., will be the piercing points of the edges of the pyramid with T.

5. Project these points, first to the *H*-projections of the respective edges, and thence to their *V*-projections. The intersection will then be the cross-hatched figures.

Note.—In case of difficulty in locating exact intersections on the *H*-projection, owing to the parallelism or "near-parallelism" of the projectors with the lines on *H*, it will be easy to locate them first on the *V*-projection, because all points projected on Q are the same distance from qq that their *V*-projections are from *GL*.

The student may argue that the foregoing illustrations are more trouble than they are worth. This may be true in the case of simple problems, such as these two are, but many difficult cases will arise where this method will be found a great saving. In all cases this method is **more exact** than the plodding method.

106. PRACTICAL EXERCISES

441. Find the length of the various rafters, MN, NK, etc., in the roof, shown in Fig. 52, and the angles they make with the horizontal. Scale $\frac{1}{8}$ in. = 1 ft.

442. Measure the angles between the rafters MN, NK, etc., and the ridges that intersect them. (Fig. 52.)

443. Measure all the valley angles between the planes A and C, E and G, etc., in the roof. (Fig. 52.)

444. A rectangular chimney 3 ft. 6 in. × 5 ft. 4 in. is to be put through the roof C in Fig. 52. Find the *H*- and *V*-projections and true size of the hole in the roof.

445. In the center of roof B in Fig. 52, a rectangular skylight 3 ft. 6 in. ×

4 ft. 6 in. is to be built. Find its H- and V-projections. In the center of roof H, a 4-ft. square opening is to be cut. Draw its projections.

446. An octagonal stone tower, 20 ft. in diameter and 20 ft. high, is to be built, surmounted by a pyramidal cap, 12 ft. high. Draw its projections, and draw in the center of each of the four visible faces of the cap one of the styles of windows shown in Fig. 53. Draw only one-half

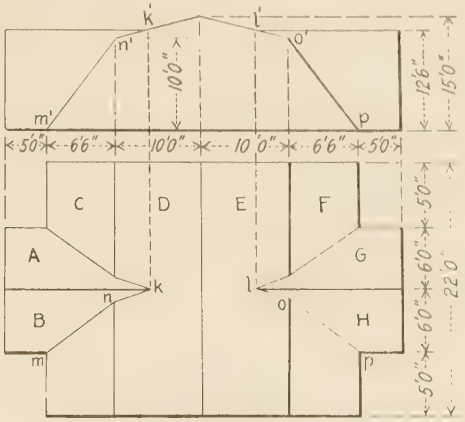


Fig. 52.

of the plan, and part of the elevation of the tower and cap. Scale $\frac{1}{4}$ in. = 1 ft.

447. A 12-in. steam feed pipe runs through a shed, its center line being A(1+1½−3) B(5+1½−½). The plane of the shed roof is T(5+3) 1(5−2). Find the projections and true size of the opening for the pipe. Scale 1 in. = 1 ft.

448. A trapeze bar, 3 ft. long, is suspended by ropes 3 ft. long. Find how high it is raised by twisting the trapeze through 90°.

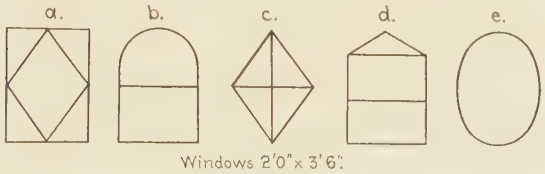


Fig. 53.

449. A conical tower is 20 ft. in diameter and 20 ft. high. Draw two lines from the apex to the base, making 45° with one another. Scale $\frac{1}{4}$ in. = 1 ft.

450. A cube of 1½″ sides is erected on H with its base diagonal || to V. Draw the traces of a plane Q || to GL, which shall cut a regular hexagon from the cube. Revolve the hexagon to prove that it is regular and *calculate* the angle that Q makes with H.

451. A square building is covered by a hip roof, which forms a regular pyramid. The hip rafters make 30° with H. What angle do the roof planes make with H? What angle do two adjacent roof planes make with one another? Use any convenient scale.

452. A 3-in. cube is drilled at two points, A on the top face $\frac{3}{4}$ in. from both the right and rear edges, and B on the left side face $\frac{3}{4}$ in. from the lower and front edges. The holes are drilled straight in at A and B respectively $1\frac{1}{2}$ in. and $1\frac{1}{4}$ in. deep. At what point on the front face, and at what angles with H and V must a hole be drilled to join the bottoms of the holes? Full size.

453. Two pulleys, 24 in.$\times$8 in., are mounted on parallel shafts. The center line of one shaft is A(1+2−3) B(2$\frac{1}{2}$+2−1), and the other center line passes through C(5+1−1$\frac{1}{2}$). The center of the pulley on AB is at B. The belt passes through a wall whose plane is T(5+3) 3(1−1). Make a rectangular opening in the wall to accommodate both belts (disregarding clearance). Draw its projections and give its true size. Draw to a scale of $\frac{1}{2}$ in. = 1 ft.

454. Inscribe a sphere in the tetrahedron A(2, 0−1) B(4, 0−1) C(3$\frac{1}{2}$, 0−3) D(3+1$\frac{1}{2}$−2).

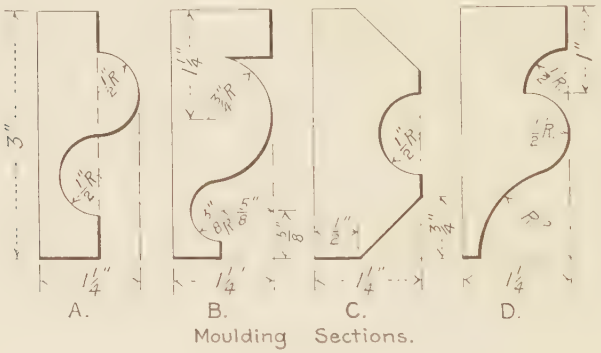


Fig. 54.

455. Inscribe a sphere in the tetrahedron E(1$\frac{1}{2}$+1−2), F(2+1−$\frac{1}{2}$) G(4+2−2) H(3+$\frac{1}{2}$−1).

456. Circumscribe a sphere about the tetrahedron M(1, 0−2) N(3, 0−1) O(4, 0−2$\frac{1}{2}$), P(2+1$\frac{1}{2}$−1$\frac{1}{2}$).

457. Find the projections of the circle passing through the points A(2+1−1), B(3+1$\frac{1}{2}$−$\frac{1}{2}$), and C(4+$\frac{1}{2}$−2).

458. Pass a plane equidistant from D(1+3−1) E(3+$\frac{1}{2}$−3) and F(4+$\frac{1}{2}$ −$\frac{1}{2}$) G(5$\frac{1}{2}$+2−3).

459. Draw the H- and V-projections of a 1$\frac{3}{4}$ in. hexagonal nut, chamfered and drilled but not threaded, having its center line K(2+2−1) L(4+1−2). Scale full size.

460. M(1+$\frac{1}{2}$−$\frac{1}{2}$) N(4+3−3) and O(3+3−2) P(5$\frac{1}{2}$, 0−1) are two pipes. What is the length of the shortest connection that can be made be-

tween them, **parallel to H,** allowing 2 in. for the connection ends. Scale 1 in. = 1 ft.

461. Determine the outline of a moulding cutter, mounted on an arbor of 3 in. diameter, to cut moulding of one of the cross-sections shown in Fig. 54. Let the center line of the arbor be 3 in. from the nearest point of the moulding. Full size.

462. Erect a pyramid on a 2-in. square in *H*, making the angle between two adjacent oblique faces 120°.

463. At a point of outcrop A(4 + 2 − 1) the strike of a body of ore is South, 50° West. The dip is 45°. A tunnel is driven from B(1 + 3 − ¼), which is on the surface, to the ore perpendicularly. Where will it strike the ore?

Note.—The "strike" of a stratum of ore is a horizontal line in the plane of the ore.

The "dip" is the inclination of the ore plane to the horizontal.

464. The point of outcrop is A(200 ft. + 200 ft. − 100 ft.), and a line is run to B, a distance from A 200 ft., South, 30° West. The altitude at B is 125 ft., and a vertical bore hole strikes ore at a depth of 25 ft. Another bore hole is made at C, on the same level as B, and 150 ft. distant from A and B. At C the hole is drilled 100 ft., striking the ore. Determine the dip and strike of the vein. Scale 1 in. = 100 ft.

465. Points D, E and F are on the side hill, which dips 20° East. D is a point of outcrop of a vein of ore. From D, E bears South 45° East 400 ft. on the slope of the hill, and F bears South 15° East 600 ft. on the slope of the hill. Vertical drill holes at E and F strike ore at 100 ft. and 150 ft. respectively. Determine the dip and strike of the vein. Scale 1 in. = 100 ft.

466. At A(8 ft. + 24 ft., 0) and B(20 ft. + 24 ft., 0) in a mill wall, two parallel 6-in. pipes run downward to the right at 45° to the floor and 30° to the wall. It is proposed to run a 12-in. cylindrical chute on the center line C(30 ft. + 16 ft., 0) D(4 ft., 0 − 24 ft.). Can this be done? If not, how much could D be shifted toward the wall to effect it? Scale ⅛ in. = 1 ft.

468. A pulley, 24 in. diameter, and 12 in. face, runs on the shaft A(2 ft. + 2 ft. − 6 ft.) B(10 ft. + 7 ft. − 2 ft.). Draw the projections of the pulley. Scale ½ in. = 1 ft.

469. Draw the projections of a 1½-in. sphere tangent to the plane T(1 + 3) 4(1 − 2) at the point O(3 + ¼, x) in T.

470. Measure the various lineal and dihedral angles of a hexagonal pyramid of 2 in. base and 2½ in. altitude.

471. One line of a roof is A(2, 0 − 2) B(4, 0 − ½). At A a rafter runs perpendicularly to AB and 60° to *H* in a northwesterly direction. At B the rafter is parallel to *V*, and the ridge is horizontal, 2 in. (scale) above *H*. Draw both projections of the roof, and find the traces of its plane. Compute its area.

472. Draw the projections of a 10-ft. circular skylight in a roof whose plane is S(8 ft. + 24 ft.) 32 ft. (8 ft. + 16 ft.), with its center at M(20 ft. + 8 ft., x) in the roof. Scale ⅛ in. = 1 ft.

473. A tunnel through a hill runs from A(425 ft.+275 ft.−100 ft.) to B(2100 ft.+150 ft.−1300 ft.). Find its length and grade.

474. A plane T passes through GL and the point K(3+1−2). A plane S parallel to it cuts V $1\frac{1}{4}$ in. above GL. Find the H-trace of S, the distance between T and S, and their angles with H and V.

475. Pass a plane through L(1+2−1) equally distant from M(4+1−$\frac{3}{4}$), N(3+2−$\frac{1}{2}$), and O(2+2−$\frac{1}{4}$). Find a point in the plane thus found which is equally distant from M, N and O.

476. Erect a cube on H with its base diagonal A(3, 0−$\frac{1}{2}$) B(4, 0−$1\frac{1}{4}$). Show its picture as seen from C($3\frac{1}{2}$+$1\frac{3}{4}$−3). Show the picture in its true size.

Note.—Pass the picture plane perpendicular to the line from the point of sight to the center of the cube.

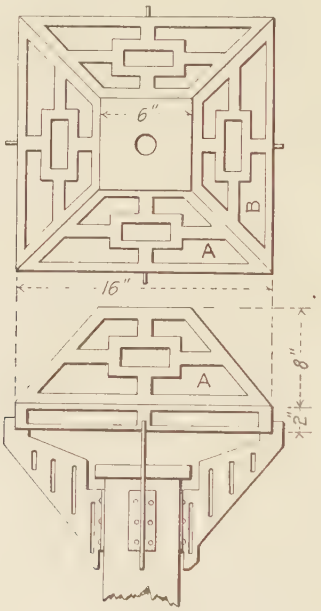


Fig. 55.—Mission lamp.

477. From a hole, A(18 ft.+10 ft., 0), in a mill wall, run a pipe that will make a straight line connection with two pipes, whose center lines are B(3 ft., 0−8 ft.) C(10 ft.+10 ft.−18 ft. 6 in.) and D(3 ft.+10 ft.−5 ft.) E(10 ft., 0−14 ft.). What are the lengths of the two portions of pipe, making no allowance for their connections? Scale $\frac{1}{4}$ in. = 1 ft.

478. Draw the projections of the circle passed through A(2+2−$\frac{1}{4}$), B($2\frac{1}{2}$+1−1), and C($3\frac{1}{2}$+2−$1\frac{1}{2}$).

479. The hinge of a door is M(4+3, 0)N(4, 0, 0). When shut, it is to the left of MN. A taut wire runs from O(3+2, 0) to P(5, 0−1). Find the outline of a slot that will allow the door to open freely through 90°

480. Measure the angles that the planes A and B, of the lampshade, shown in Fig. 55, make with one another, and with H and V.
481. Make a full set of details to scale of all the glass and sheet iron pieces in the lamp shade and supports, shown in Fig. 55. Do not duplicate the details, but state how many are required of each. Allow for the folding of the iron pieces.
482. Make a detail of the backbone of the saw horse shown in Fig. 56, showing the angles of the cuts to be made. Draw to any convenient scale.
483. Make detail drawings of the legs and braces of the saw horse, shown in Fig. 56, showing the angles of the various cuts. Draw to any convenient scale.

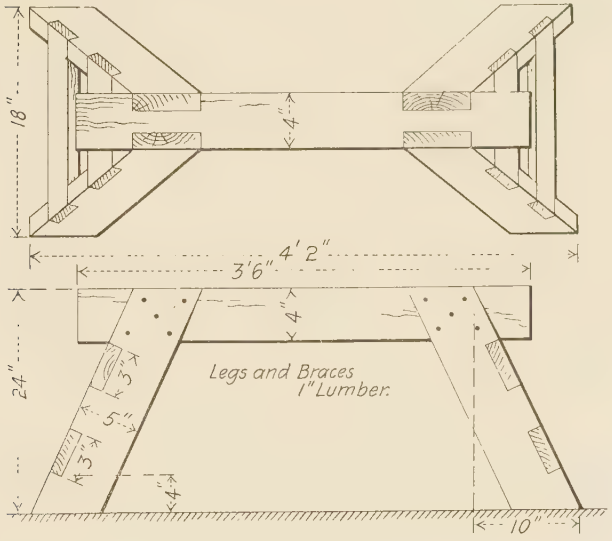


FIG. 56.—Saw horse.

484. Make a pattern to any convenient scale of the grain hopper, shown in Fig. 57. Use two spaces for the projections and pattern. Make a paper model of the hopper, putting on a flap for holding together.
485. Make a pattern to any convenient scale, of the ventilator cap, shown in Fig. 58.
486. Let H be a mirror. A ray of light from A$(1+2-2)$ passes through B$(3+\frac{1}{2}-1)$. At what point does it strike V?
487. Find the path of a ray of light that passes through M$(2+3-2)$ and is reflected from H and V to N$(4\frac{1}{2}+2-\frac{1}{2})$.
488. The plane of the top of a bin door is T$(1+2)3\frac{1}{2}(1-3)$. The center line of a 12-in. square chute to the bin runs from O$(4+3,\ 0)$ runs at an angle of 30° to H and 30° to V. Find the opening in the bin, and its true size. Scale to suit.

489. Fig. 59 shows a stool, or plant stand, made of 1-in. lumber. The edges of the four inclined supporting pieces are chamfered to fit. Make a working drawing with full details to scale, showing the angles of the champered pieces, without duplicating any of the parts. Take two spaces for the work.

490. Draw the projections of four spheres of 1 in., $1\frac{1}{4}$ in., $1\frac{1}{2}$ in., and $1\frac{3}{4}$ in. respective diameters, each sphere tangent to the other three.

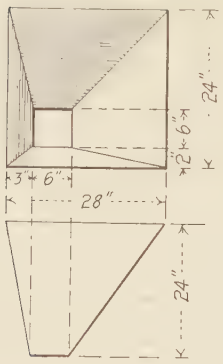


Fig. 57.—Grain hopper.

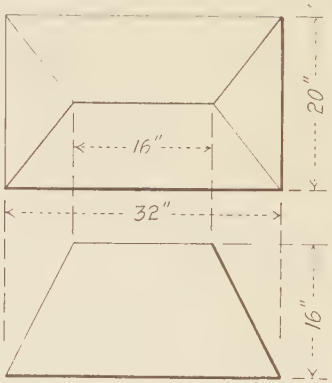


Fig. 58.—Ventilator cap

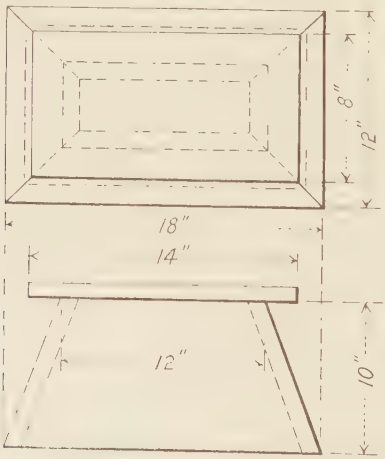


Fig. 59.—Stand.

491. Develop the surface of the keystone of the arch shown in Fig. 60. The masonry is 12 in. thick. Use any convenient scale.

492. A borehole cuts a 6-ft. core from a vein, whose strike is North 60° East, and whose dip is 45° to the Northwest. The borehole bears South 75° West and dips 30°. What is the real thickness of the vein?

493. At A, where the elevation is 9800 ft., a vertical borehole strikes a vein at 300 ft. B bears North, 30° East, 1500 ft. from A, and has an elevation of 9600 ft. An inclined hole is drilled, bearing North 75° East, dipping 75°, striking the vein at 525 ft. At C, whose elevation is 9900 ft., 1200 ft. from A, South 60° East, a vertical bore hole cuts the ore at 375 ft. Required the dip and strike of the vein. Let H be at 9500 ft. level.

494. At A(1+3−3) a borehole is drilled, bearing North 30° East, and dipping 75°. It intersects a vein, whose strike is due North, and cuts an 8-ft. core. The true thickness of the vein is 5 ft. What is the dip of the vein?

495. Vertical holes on a hill side (considered a plane) are bored 150 ft. at A(100 ft.+400 ft.−300 ft.), 200 ft. at B(350 ft.+250 ft.−325 ft.), and 125 ft. at C(600 ft.+325 ft.−100 ft.), striking a vein of ore. What is the line of outcrop, and what are the dip and strike of the vein?

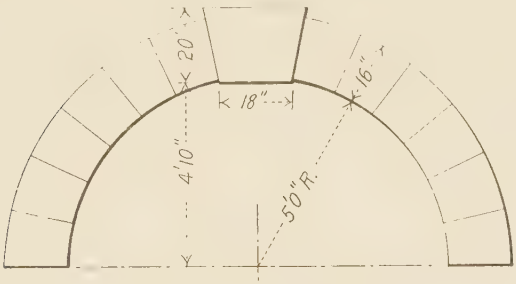


FIG. 60.—Stone arch.

496. Find the line of intersection between the square pyramid A(3, 0−$\frac{3}{4}$) B(4$\frac{1}{4}$, 0−2) C(3, 0−3$\frac{1}{4}$) D(1$\frac{3}{4}$, 0−2) E(3+2$\frac{1}{2}$−2), and a triangular prism, its lower face parallel to H, its center line M(2+1−1) N(5+1−2), and its cross-section an equilateral triangle of 1-in. sides.

497. Find the line of intersection between a hexagonal prism of 2-in. base and 2$\frac{1}{2}$-in. altitude, standing on H, and a 1-in. square prism, whose axis is parallel to H, 1 in. above H and $\frac{1}{4}$ in. away from the center line of the hexagonal prism.

498. Erect a square prism with its base on H, of 1$\frac{1}{2}$-in. base and 2-in. altitude. Make the base diagonal parallel to V. Intersect by a hexagonal prism, 1$\frac{1}{4}$ in. long diameter, with two faces parallel to V. Let the two center lines intersect at an angle of 60° at a point 1 in. above H. Find the line of intersection.

499. Erect a 2-in. cube on H. Draw a 1-in. triangular prism, whose center line passes through the upper, rear, right-hand corner of the cube, making angles of 45° with H and 30° with V, running through the cube. Find the intersection of the solids.

500. A(2+1−2) B(4+2−1) is the center line of a 4 in.×12 in. hollow tile, of square section with rounded corners. The walls are $\frac{1}{2}$ in. thick,

and the radius of the outside rounded corners is 1 in. Draw its projections quarter size.

SHADES AND SHADOWS

107. One of the most interesting applications of the elementary work in Descriptive Geometry is found in obtaining the shades and shadows of objects. Naturally, this application makes its greatest appeal to the architectural student, but no engineer should be without knowledge of its workings, although he may rarely make professional use of it.

The distinction between **Shades** and **Shadows** is determined by their definitions.

Shade is that part of an object not exposed to the direct rays of light.

Shadow is that portion of a **surface in light** from which rays of light are excluded.

Umbra is the unlighted space behind an object in light.

Note.—Shadow is the intersection of the umbra with a surface in light. That is, it is the interrupted portion of an umbra by an interposed surface.

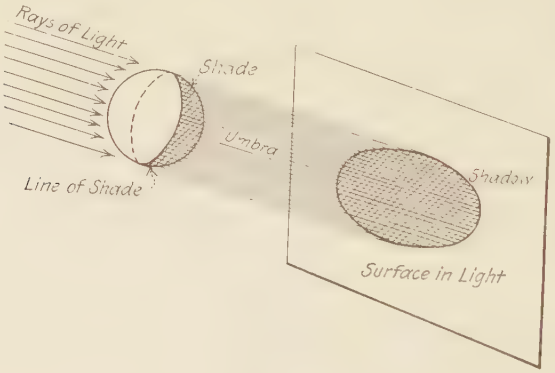


Fig. 61.

The **Line of Shade** is the imaginary line between the light and shaded portions of an object.

The picture (Fig. 61) shows the distinctions between these characteristics of light effects.

The sun is our source of light, and its rays are assumed to be straight lines all parallel. This assumption is very nearly true, and the deviation of the rays from parallelism is so very slight that no account need be taken of the error. For convenience in

conventional shading, the sun's rays are always assumed to be passing over the left shoulder of the observer in the direction of the diagonal of a conventional cube; that is, both the H and V projections of the rays make 45° with GL.

Referring to Fig. 61, we have the **rays of light** depicted as coming toward a sphere in parallel lines, and either stopping, if they strike the sphere, or passing on, if they do not. One-half of the sphere is in **light,** and one-half in **shade,** and the **line of shade** is a great circle of the sphere. This determines the **umbra** to be a cylinder of diameter equal to the diameter of the sphere. The umbra, on meeting another surface, is interrupted, and its intersection with that surface is the **shadow.**

108. Shadows made up of piercing points of rays.

From the foregoing, it will appear that the integral of the tangent rays (the tangent surface) to the sphere will pierce the **surface in light** in the **outline of the shadow.** Hence, the finding of the **line of shade** resolves itself into finding the tangent rays of the object, their points of tangency making up the **line of shade.** These tangent lines form the **surface of the umbra,** and their piercing points with the **surface in light** form the **outline of the shadow.**

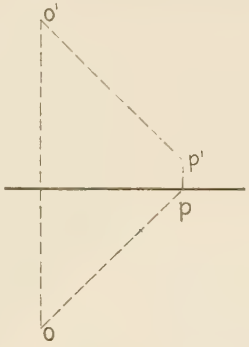


FIG. 62.
The shadow of a point.

The following rules are thus deduced:

109. The **Shadow** of a **Point** is a **Point.**
The **Umbra** of a **Point** is a **Line.**

110. The **Shadow** of a **Line** is, in general, a **Line.**
The **Umbra** of a **Line** is, in general, a **Plane.**

111. The **Shadow** of a **Plane Figure,** cast on a plane, is, in general, a **Plane Figure** of similar outline.
The **Umbra** of a **Plane Figure** is, in general, a **Prism.**

112. The Method of Finding Shadows.

Fig. 62 shows a point O in space, acted upon by a ray of light in the conventional direction, OP. The V-piercing point of the ray OP is P, which is the shadow of O on V. Fig. 63 shows a line AB in space. From A and B draw rays of light AC and AD, having their H- and V-projections 45° from GL. Find their H-piercing points, C and D. The shadow, therefore, of AB on

H is CD. The umbra of the line is the plane ABCD, and its H-trace is the shadow, CD.

Note.—In case one end of the line casts its shadow on H, and the other end casts its shadow on V, the two traces of the umbra plane will be the shadow, and it will be a broken line, partly on H, and partly on V. This condition will be found later in Fig. 66.

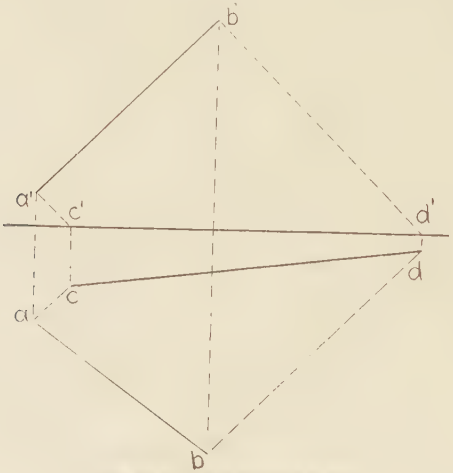


FIG. 63. The shadow of a line.

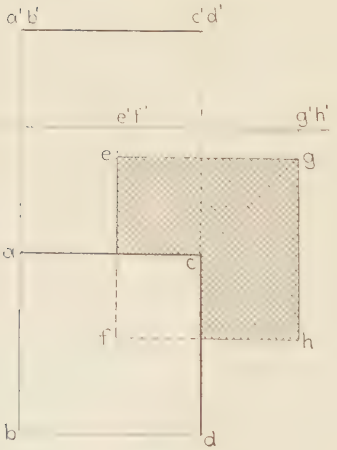


FIG. 64.—The shadow of a plane figure.

To find the shadow of a square (or other plane figure).

Let ABCD (Fig. 64) be a square, and let AE, BF, CG and DH be the rays passing the four corners. The H-piercing points

of the four rays are E, F, G and H, and the quadrilateral formed by joining them in order is the shadow of the square on *H*.

Note.—If the square is parallel to *H* (as in this figure) the shadow will be an equal square. Prove this statement.

To obtain the shadow of a cube (or other solid figure).

Draw the projections of a cube, such as is shown in Fig. 65.

From B, C and D draw rays of light, piercing *H* in the points E, F and G.

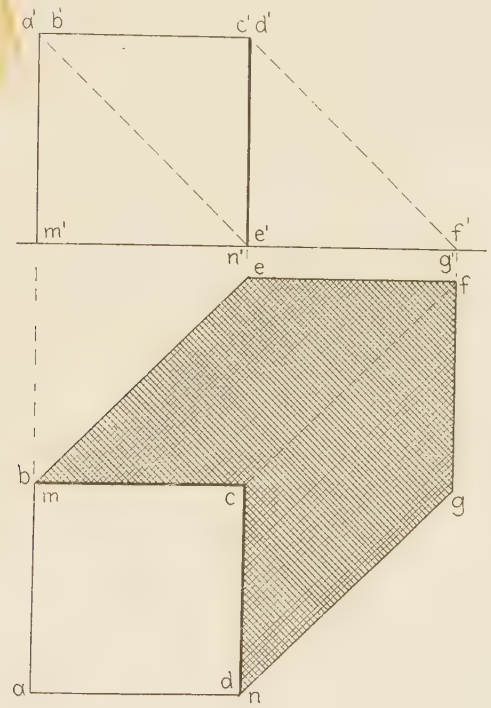


FIG. 65.—The shadow of a cube.

The line of shade follows this course: MBCDN. The line MB casts the shadow ME, the line BC casts the shadow EF, the line CD casts the shadow FG, and the line DN casts the shadow NG. This gives us the entire space, mefgncm. as the shadow of the cube on *H*. No ray runs past A, because it is on the light surface. A ray through A would penetrate the cube, and would merely be inside the umbra, and would pierce the surface in light within

the shadow, and not on the outline. In shadows, only points on the outline are necessary to be found.

To obtain the shadow of an object when it falls on more than one plane.

Let it be required to draw the shadow of the hexagonal prism shown in Fig. 66.

The procedure in this case, as in all cases, is to draw rays from the corners of the object, and find their piercing points with the first object they strike.

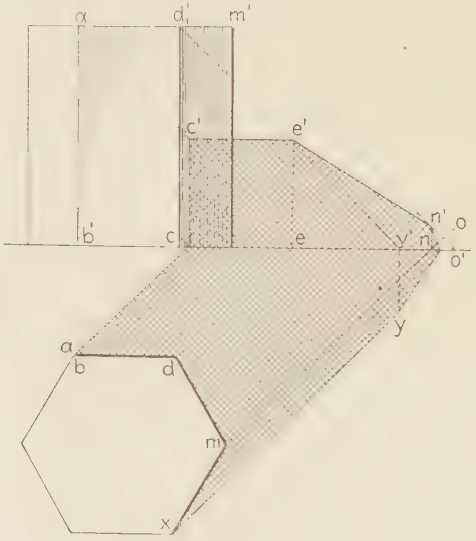


FIG. 66.—The shadow of a solid falling on two planes.

In this figure all the rays but that from X pierce the V-plane first, yet the object is some distance from V, and, therefore, a portion of the shadow must be on H.

The ray from A pierces V at C, but the shadow of the line AB is partly on V and partly on H. The shadow of AB is drawn on H until it intersects GL, then from that point it is joined with the V-piercing point of AC. Thence the outline is drawn through the V-piercing points of the rays from D and M, respectively E and N. The line MX casts a shadow on both V and H. The piercing point of the M ray is at n′ and that of the X ray is at y. The shadow of MX will be a broken line. Taking any intermediate point on the line, and drawing from it a light ray, and finding its piercing point, will determine the direction of

its shadow on either plane. The easiest way is to find the *H*-piercing point of the M ray at O. The shadow of MX on *H* will then be the line yo. From the point where yo crosses *GL*, the shadow is interrupted by *V*, so connect that point with n′, and the shadow is complete.

The foregoing illustrations furnish the rudiments of "Shades and Shadows." By working the following exercises, you will obtain a fair knowledge of the subject, as the most complicated work of the kind is merely an elaboration of these principles.

EXERCISES

113. In these exercises use double-cross-hatching for the shadows, and single for the shades, and dotted lines for the invisible part of the shadow, as in Fig. 66.

501. Erect a cube on *H*, having A(3, 0−1) B(3, 0−3) for its base diagonal. Determine its shades and shadows.

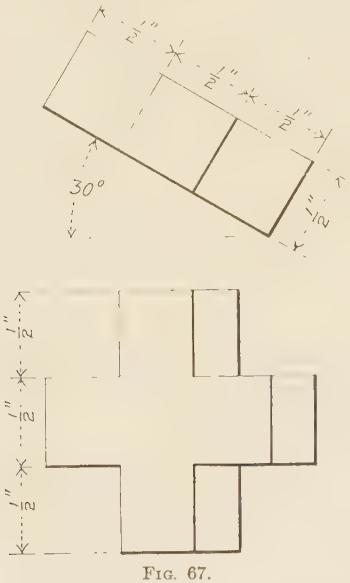


Fig. 67.

502. Find the shades and shadows of an oblique cone, whose axis is C(2+0−2) D(4+2−1), and whose base is a 1½ in. circle in *H*.

503. Let E(3, 0−1½) F(3+3−1½) be the center line of a hexagonal pyramid of 2 in. base. Find its shades and shadows.

504. Let G(1½+¾−1½) H(4+¾−2) be the center line of a hexagonal prism, with one face on *H*. Draw its shades and shadows.

505. Find the shades and shadows of a 2-in. cylinder, whose axis is M(2,0−2) N(2+3−2).

506. Find the shades and shadows of a cylinder lying on H, whose axis is O($1\frac{1}{4}+\frac{3}{4}-1$) P($3+\frac{3}{4}-2$).
507. Find the shadow on H and T($5+2$)1($5-2$) of a 1 in. square prism, 2 in. high, having its base center at A(2, $0-2$).
508. Find the shadow on H and T($5+3$)3($3-3$) of a hexagonal pyramid, of $1\frac{1}{2}$ in. base and 2 in. altitude, the center of its base being B(2, $0-2$).
509. Find the shades and shadows of a 1 in. square prism, 2 in. high, surmounted by a $1\frac{1}{2}$ in. square plinth, $\frac{1}{2}$ in. thick. The center line of both solids is A($1\frac{1}{2}$, $0-2$) B($1\frac{1}{2}+2\frac{1}{2}-2$).
510. Make the prism in Exercise 509 a 1 in. cylinder, and the exercise otherwise the same.
511. Find the shades and shadows of a cross with the dimensions and position shown in Fig. 67. Place the center of the base at A(2, $0-1$).

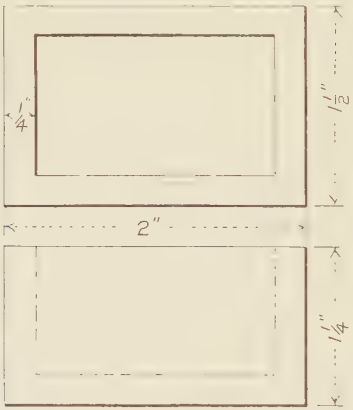


Fig. 68.

512. Place the cross having the same dimensions and position as that given in Fig. 67, with its center at B(2, $0-2$). Find its shades, and its shadow on H and T($4+2$)1($4-2$).
513. Draw a stepped pyramid, made up of four plinths, each $\frac{1}{4}$ in. thick, and $1\frac{3}{4}$ in., $1\frac{1}{2}$ in., $1\frac{1}{4}$ in., and 1 in. square respectively. Draw the cross of dimensions and position shown in Fig. 67 on the top of the pyramid. Obtain the shades and shadows. Draw the stepped pyramid well to the left of the drawing, and "square" with V.
514. Place a prism, whose right section is a $1\frac{1}{4}$ in. equilateral triangle, $1\frac{1}{2}$ in. long, on a 1 in. cube (so as to resemble a house and roof), and find their shades and shadows.
515. Draw the box (Fig. 68) so that one of its long sides is 30° from V, and draw its shades and shadows, inside and outside, and on H and V.
516. Place the box (Fig. 68) on the plane S($1+1$)3($3-3$), and find its shades and shadows, inside and outside, on H, V and S.
517. Place the box (Fig. 68) on end with the bottom toward and parallel to V, and draw its shades and shadows.

CHAPTER III

CURVED LINES AND SURFACES

114. Definitions.

A **Line** is the path of a point.

A **Straight Line** is a line in which the point moves continually in one direction.

A **Curved Line** is a line in which the point moves in a direction that is being changed constantly in obedience to some law.

A line is regarded as having one dimension only—length, without breadth or thickness. It is considered as being made up of infinitely small points, separated by infinitely small spaces. The line may, therefore, be regarded as being generated by an **immaterial** point, occupying **consecutive positions.** Two of these consecutive points determine a straight line, of infinitesimal length, but of a **determinate direction.** Such a line is called an **Element;** that is, it is one of the innumerable infinitely small straight lines that compose every line, straight or curved.

115. Classification of Lines.

Lines are **Straight** or **Curved.**

Curved Lines may be **Plane Curves,** also called **Lines of Single Curvature,** or they may be **Space Curves,** also called **Lines of Double Curvature.**

A Plane Curve is a line moving always in a plane, changing its direction in accordance with some law. Examples, circles, ellipses, etc.

A Space Curve is a curved line, no four consecutive points of which are contained in any plane. Example, the helix.

Table of Lines

Straight.	Only one kind.	
Plane Curves	Circle	Conic Sections.
	Ellipse	
	Hyperbola	
	Parabola	
	Involutes	Gearing Curves.
	Cycloidal Curves	
	Spirals, etc.	
Space Curves	Helix	
	Most Intersections of Curved Surfaces	
	Spherical Epicycloid, Spherical Hypocycloid.	

116. Projections of Curves.

A curve is projected on H, V and P in the same manner as a straight line; that is, all the points in the curve are projected on the plane of projection in lines perpendicular to the plane, and the sum of these projections of points makes up the projection of the line. As these projectors are all parallel, the surface which they compose is called the **Projecting Cylinder** of the curve. (Compare with the projecting plane of a straight line.)

Usually a small number (six or eight) points in space will very accurately determine a curve. Its projections are drawn by running a smooth line through the projections of the points in order, on both H and V, and on P, if necessary.

Note.—Plane curves, in certain positions, will project on one plane as a straight line, and the projecting cylinder is in that case a projecting plane. However, most curves do not lie so conveniently, and space curves **never** project as a straight line. Any curve, that is parallel or oblique to the plane of projection, will project as a curve on that plane. Any plane curve, lying in a plane perpendicular to the plane of projection, will project as a straight line on that plane.

117. Plane Curves lying in a Profile Plane.

No curve, lying in a profile plane, can be shown in its true character by the H- and V-projections, because they will both be straight lines. The P-projection will show, not only the true character of the curve, but its true dimensions. The H- and V-projections of a number of points in a curve will **determine** it, but will not **show** it.

In a similar way, any curve, lying in a plane parallel to H, is projected in its true size on H, and as a straight line on V.

If the curve lie in a plane parallel to V, its V-projection gives its true size, and its H-projection will be a straight line.

If a curve lie in a plane perpendicular to H and oblique to V, its H-projection will be a straight line, and the V-projection will be curved, and will show the character of the curve. The curve, however, may be entirely changed by this foreshortening; *e.g.*, the projection of an oblique circle is an ellipse.

The H- and V-piercing points of a curve are found by the same rule that obtains for straight lines; that is, where the V-projection crosses GL, project to the H-projection to find the H-piercing point, etc.

THEOREMS

Thorem XXIX.—Any curve in a plane perpendicular to a plane of projection is projected thereon as a straight line.

Theorem XXX.—Any curve lying in a plane parallel to a plane of projection is projected thereon in its true size.

Theorem XXXI. -No space curve is ever projected as a straight line.

118. Tangents and Normals to Curves.

Definition.—A **tangent** to a curve is a line containing two consecutive points of the curve.

The tangent may be straight or curved, but, when curved, it is always especially designated. When spoken of as a tangent, the understanding is that it is a **straight line tangent.**

119. The tangent (from its definition) must lie in the plane of the curve at the point of tangency. Also (from its definition) it must have the same direction as the curve at the point of tangency. It is especially important to remember that the tangent must lie in the plane of the curve, as many errors appear from disregarding this fact.

Note.—The apparent discrepancy between the foregoing definition of the tangent and that given in most Plane Geometries amounts to nothing. They define a tangent as a line touching a curve at one point. Their definition answers sufficiently for beginners in Geometry, and for a section of the subject that confines its operations to a single plane, but the student can easily see how such a definition would be utterly inadequate in Space Geometry.

Note.—A curve that is tangent to a curve may contain more than two consecutive points of the line to which it is tangent. It brings about no new relations by reason of this fact, and such a condition may be overlooked.

120. As the tangent is a short straight line in the curve (two consecutive points), the curve may be considered as being composed of an infinite number of infinitesimal tangents, intersecting consecutively. Fig. 69 shows a series of straight lines, A, B, C, etc., intersecting at short finite intervals. They compose so nearly a curve, that it is easy to imagine that minute divisions would compose an actual curve. From the figure it can easily be seen how the tangent at any point determines the direction of the curve at that point.

Theorem XXXII.—A tangent to a straight line is the line itself.

Theorem XXXIII. At any point on a curve, one straight line tangent and an infinite number and variety of curved line tangents may be drawn.

Theorem XXXIV. –All of these tangents will also be tangent to each other.

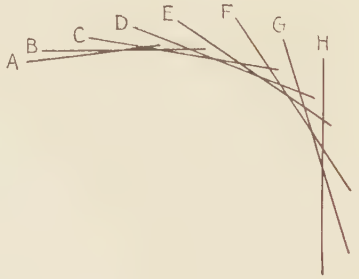


FIG. 69.

121. Projections of the Tangent.

The projections of the tangent to a curve will be tangent to the projections of the curve on the respective planes. The converse is not necessarily true, because a line lying in a different plane from that of the curve may have its projection tangent to that of the curve on one plane, **but not on any other.** So we have the following theorem.

Theorem XXXV.—If two projections of a line and a curve are tangent, the line and curve are tangent.

122. To Draw a Tangent to a Curve.

There are exact and approximate methods. In the case of the circle, the tangent is always perpendicular to the radius at the point of contact. In the case of other conic sections, the exact method will be taken up in connection with the study of those curves. For ordinary drafting work a sufficiently accurate tangent may be drawn by placing a straight edge against the curve, and laying off the tangent with the eye. A simple approximation, that can be used in nearly all cases of curves, and is exact

for the circle, is this: Let it be required to draw the tangent at A (Fig. 70) on the curve shown. Set the dividers at some small arbitrary span, and lay off B and C on the curve at equal distances from A. Connect B and C with a straight line. Through A draw AD (the tangent) parallel to BC.

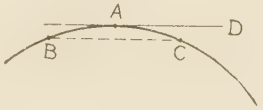


Fig. 70.—Drawing a tangent.

This method will not do for curves changing their radii rapidly. The eye will do close work in sharply curving lines. Numerous close approximations may be found in other text-books, and an especially good one is given in MacCord's "Elements of Descriptive Geometry," page 93, but its **practical** value is small.

123. Normals.—A normal to a curve at any point is a perpendicular to the tangent at the point of tangency. While any perpendicular is normal to the curve, it is customary to regard the normal as the perpendicular to the tangent in the plane of the curve.

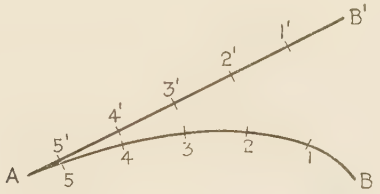


Fig. 71.—The rectification of a curved line.

In the case of a space curve, the normal may be any line perpendicular to the tangent at the point of tangency.

124. The Rectification of Curves.

To rectify a curve means to straighten it out; that is, to draw a tangent at any point equal in length to any given portion of the curve.

First.—**To rectify a portion of any curve.**

Let the curve be AB, Fig. 71. Set the dividers conveniently small, so as to lay off an arc, whose chord is approximately the length of the arc. Step off with the dividers the points 1, 2, 3, 4,

etc., from B toward A, where the tangent is drawn. Lay off with a different pair of dividers the last step, 5A (which is not likely to be the same as the others), on the tangent A5′. Then, with the original dividers, step off from that point the points 4′, 3′, 2′, 1′ and B′. This rectification is fairly good, if care is taken.

Second.—**To rectify a small arc of a circle.**

If the arc be less than 60°, a convenient method is shown in Fig. 72. Let AB be the arc, and BD (of indefinite length) be the tangent. Extend the chord AB through B one-half its length to C. Using C as a center, and CA as a radius, strike the arc AD. BD will then equal (approximately) the arc BA.

Third.—**To rectify a larger arc.**

Divide the large arc into smaller ones, rectify each, and add them together.

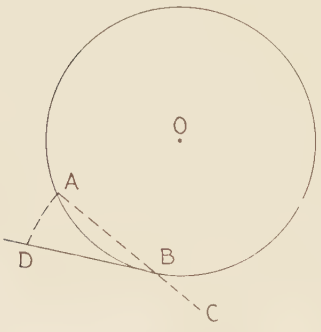


Fig. 72.—The rectification of a circular arc.

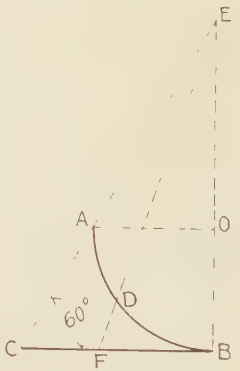


Fig. 73.—The rectification of a quadrant.

Fourth.—**To rectify a quadrant of a circle.**

Let AB (Fig. 73) be the given quadrant. Draw the tangent at B, and from A draw a line AC, which makes 60° with CB. BC is then a very close approximation to AB. This is undoubtedly the closest approximation for such an arc. Let the student prove by trigonometry that it is numerically equal to $\frac{\pi}{2}$ R. Although there is no geometrical proof for this, your calculation will show that it is closer than the accuracy of your instruments.

A convenient method of rectifying a small arc comes with

this method. To rectify BD, Fig. 73, draw the line ED through to the tangent at F. The line BF equals BD.

Fifth. —**To rectify an arc of a circle when its angle is known.**

Here the numerical method is best. The arc AB (angle = θ) $= 2\pi R\frac{\theta^\circ}{360^\circ} = .01745R\theta^\circ$

125. To lay off any length of line, straight or curved, on any curve.

Regard the line AB′ (Fig. 71) as a curved line. Lay off AB on the curve in exactly the same manner that it was laid off on the straight line.

This is often a necessary operation in practical drafting; example, in drawing cycloids for gear outlines.

126. Conic Sections.

Definitions.

Circle.—A circle is a plane curve generated by a point moving so as to be at a constant distance from another point.

Ellipse.—An ellipse is a plane curve generated by a point moving so that the **sum** of its distances from two fixed points in its plane is constant.

Hyperbola.—A hyperbola is a plane curve generated by a point moving so that the **difference** between its distances from two fixed points in its plane is constant.

Parabola.—A parabola is a plane curve generated by a point moving so that its distances from a **line and a point** in the plane are always equal.

The four foregoing curves are called "Conic Sections" from the fact that they are the possible curves that can be cut from a right circular cone by a plane. A plane can also cut a straight line from a cone. The necessary conditions for cutting these lines from a cone are:

Straight Line.—Plane through the apex.

Circle.—Plane perpendicular to the axis.

Ellipse.—Plane, oblique to the axis, cutting all the elements on one side of the apex.

Hyperbola.—Plane, parallel to the axis, or a plane oblique to the axis at a smaller angle than that made by the elements of the cone.

Parabola.—Plane parallel to one element of the cone.

127. Gearing Curves.

Cycloid.—A cycloid is a plane curve generated by a point on

the circumference of a circle, as the circle rolls on a straight line.

Epicycloid.—An epicycloid is a plane curve generated by a point on the circumference of a circle, as it rolls on another circle.

Hypocycloid.—A hypocycloid is a plane curve generated by a point on the circumference of a circle as it rolls on the inside of another circle.

Involute.—An involute is a plane curve generated by a point on a tangent, as the tangent rolls on a plane curve or polygon. The involute of a circle is the only one of any considerable importance, and is the curve used in gearing.

The foregoing are called "gearing curves," because they are mainly used as the outlines of teeth in gear wheels. The outlines of bevel gears are (theoretically) spherical cycloids, but their consideration is negligible in this work, and practically so in actual drafting work.

SPACE CURVES

128. Helix.—A helix is a curve generated by a point moving about a straight line at a constant distance, and in the direction of the line at a constant velocity.

Note.—This line is the line of the screw thread, twist drill, screw conveyor, worm gear, coiled spring, and many other important practical devices. It is a curve on the surface of a cylinder.

There is also a **conical helix,** that sometimes appears in practical work. It is usually drawn so that it makes one complete revolution about the axis in travelling from the base to the apex, ascending the surface at a uniform rate of motion. The plan of the conical helix is the Spiral of Archimedes. By drawing the plan first and projecting the points of the curve as it intersects the various elements of the cone, the elevation is easily drawn. It is rather commonly used in sheet metal work and in spiral springs.

There is no need of defining or considering any other of the space curves.

GENERAL DEFINITIONS

129. Axis.—The line about which a point revolves, as in the case of the helix. It is also a center line for some curves, as the ellipse, parabola, and hyperbola.

Minor Axis and **Major Axis.**—In the ellipse there are two dimensions, sometimes called the long and short diameters, but more generally the major and minor axes. They are respectively the longest and shortest possible straight lines running through the center from one point in the curve to another. The **constant sum,** spoken of in the definition, is equal to the length of the major axis.

Focus.—The two fixed points, mentioned in the definitions of the ellipse and hyperbola, and the single one, mentioned in the definition of the parabola.

Vertex.—The nearest point in the curve to the focus. It has the smallest radius of curvature of any point in the curve, and might be described as the center of symmetry of the curve. The circle has no vertex.

The **constant difference,** mentioned in the definition of the hyperbola, is the distance between the two vertices of the curve.

Directrix.—The straight line mentioned in the definitions of the parabola and the cycloid.

Generating Circle.—The rolling circles in the definitions of the cycloid, epicycloid and hypocycloid.

Pitch Circle, or **Directing Circle.**—The circle on which the generating circle rolls.

Evolute.—The curve on which the tangent generating the involute rolls.

Note.—The tangent to the evolute is normal to the involute.

Pitch.—The pitch of a helix is the distance that it travels in the direction of the axis in making one complete revolution.

PROBLEMS RELATING TO CURVED LINES

130. Problem 44.—To draw an Ellipse, having given its foci and "constant sum."

Construction.—1. Draw the line AB (Fig. 74) and from a center, O, lay off F and F′ at equal distances from O, making FF′ equal to the given distance between the foci.

2. On the same line lay off A and B at equal distances from O, making AB equal to the given constant sum.

3. Spread the compasses equal to AO, and strike arcs from F and F′ as centers. The intersections, C and D, of these arcs are the ends of the minor axis.

4. Set the compasses at arbitrary radii, A1, A2, etc., and strike arcs with them from F and F′ as centers.

5. Set the compasses at complementary radii, B1, B2, etc., and strike arcs from F and F′ as centers.

6. Through A, B, C, D, and the intersections of complementary radii, draw a smooth line. This will be the required ellipse.

Proof.—The sum of the distances of any of the points, thus found, to the foci is always equal to AB.

Note.—There are numerous ways of drawing an ellipse, exact and approximate, which are found in any text-book on Mechanical Drawing. Probably the quickest and most convenient is the "trammel method," and next easiest is the "concentric circles method." Any method whereby the compasses are used to describe the curve is necessarily inexact, because no arc of a circle is any arc of an ellipse. However, in practical work, it is often expedient and desirable to use such ellipses, especially in Isometric Projection.

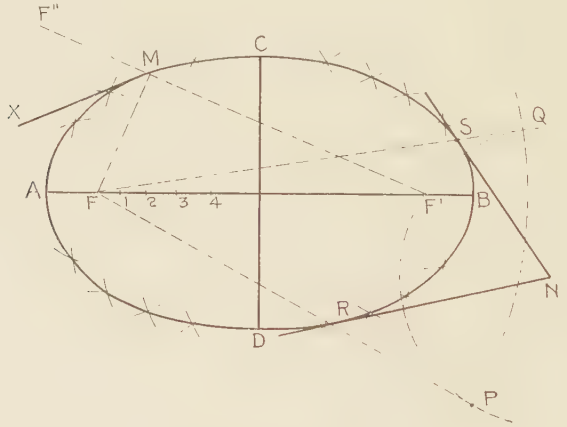


FIG. 74.—The ellipse.

131. Uses for the Ellipse.—In ornamental work the ellipse is extensively used, in its entirety, and in portions. The ellipse is the projection of any circle in an oblique plane, and its major axis is always equal to the diameter of the circle. In machine design, the ellipse is used in certain "quick return" motions.

Limits.—When the distance between the foci is zero, the ellipse becomes a **circle**. When the distance between the foci equals the "constant sum," the ellipse becomes a **straight line.**

132. Problem 45. To draw the tangent to an Ellipse at a given point on the curve.

Construction.—Let M (Fig. 74) be the point.

1. Draw MF and MF′, the focal lines.
2. Extend MF′ beyond M to F″.
3. Bisect the exterior focal angle FMF″.

The bisector MX will be the tangent at M.

133. Problem 46. To draw the tangent to an Ellipse from a point outside.

Let N (Fig. 74) be the point.

1. Draw a circle with N as center and NF′ as radius.
2. Intersect the circle in points P and Q by an arc whose radius is equal to AB, and whose center is at F.
3. Connect either P or Q with F.
4. Where PF or QF intersect the ellipse at R or S, will be the points to which tangents can be drawn from N.

134. If the axes of an ellipse are given, the foci may be determined by striking an arc from C or D as center, with a radius equal to AO. Where this arc intersects AB will be the foci.

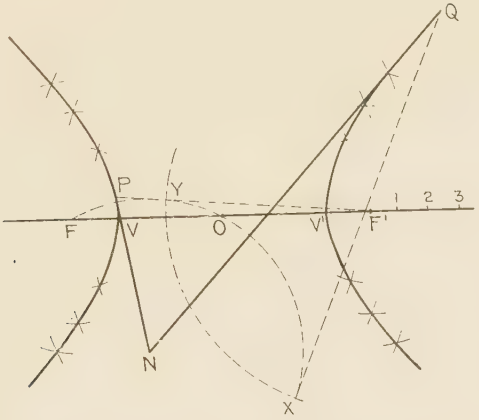


Fig. 75.—The hyperbola.

135. Problem 47.—To draw a Hyperbola, having given its foci and the "constant difference."

Let F and F′ (Fig. 75) be the foci, and the "constant difference" be equal to the distance VV′.

Construction.—1. Lay off F and F′ and V and V′ on the same line, at their proper distances, all symmetrically placed about a center, O. (V and V′ will be the vertices.)

2. Spread the compass at any arbitrary spans, as V1, V2, etc., and strike arcs with these radii with F and F′ as centers.

3. Using the complementary radii, V'1, V'2, etc., strike arcs with F and F' as centers, intersecting those already drawn.

4. Draw a smooth curve through the proper intersections and V and V'. This will be the required hyperbola.

Note.—The hyperbola is in reality two curves, each having a vertex, and each unlimited in extent. The hyperbola varies in accordance with the ratio between FF' and VV'. The most important use of the hyperbola is that it is the generating line of the pitch surface of "skew" gears.

136. Problem 48.—To draw the tangent to a Hyperbola from any point of the curve.

Construction.—Let P be the point.

1. Draw PF and PF', the focal lines.

2. Bisect the angle FPF'.

The bisector will be the tangent.

Note.—The construction is not shown in the figure. Let the student make it for himself.

137. Problem 49.—To draw a tangent to a Hyperbola from any point outside the curve.

Let N (Fig. 75) be the given point.

Construction.—1. With N as a center, draw a circle passing through F', the nearest focus.

2. With F, the farther focus, as the center, and VV' as the length of the radius, strike an arc, cutting the first circle in the points X and Y.

3. Draw XF and YF, extending them until they intersect the hyperbola in the points P and Q.

4. NP and NQ will be the required tangents.

138. Problem 50. To draw a Parabola, having given the focus and the directrix.

Construction.—Let XD (Fig. 76) be the directrix and F the focus.

1. Draw a line CG through F perpendicular to XD. This is the axis of the curve.

2. Bisect the distance from F to XD at E. E will be the vertex.

3. Draw a number of lines at points 1, 2, 3, etc., parallel to XD, at arbitrary distances.

4. Locate two points on each of the lines, equally distant from F and XD.

5. Draw a smooth line through the points thus found, and it will be the required parabola.

139. Problem 51.—To draw the Tangent to a Parabola at any point on the surface.

Construction.—Let M (Fig. 76) be the given point.

1. Draw MX, perpendicular to XD.
2. Draw the focal line MF.
3. Bisect the angle FMX. The bisector is the tangent.

Note.—Extend the line XM in the direction MY, and draw the normal MO. Notice that MY is parallel to the axis, and that MO is the bisector of the angle FMY, making OMF equal to OMY. This fact is taken advantage of in designing the reflector for the searchlight. M is a random point on the parabola, and shows that **any ray** from F, striking the parabola at M (**any point**) is reflected in a line parallel to the axis, as MY. This concentrates all the reflected rays into a cylinder of light, preventing diffusion.

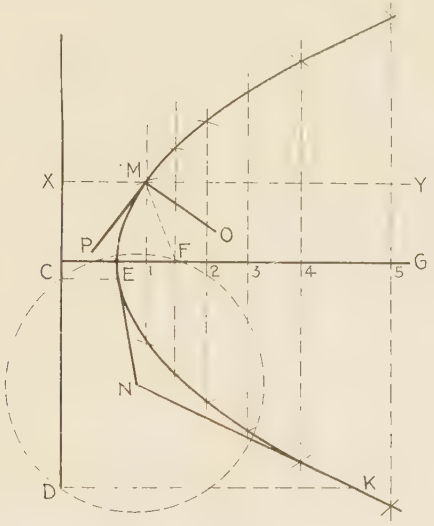


Fig 76.—The parabola.

140. Problem 52.—To draw a Tangent to a Parabola from a point outside.

Construction.—Let N (Fig. 76) be the given point.

1. With N as a center, draw a circle through F.
2. From its intersections, C and D, with DX, draw the perpendiculars, CE and DK.
3. E and K, the intersections with the curve, are the points to which tangents can be drawn.

141. Problem 53. To draw a Cycloid, having given the directrix and generating circle.

Construction.—Let AB be the directrix and O be the center of the generating circle, which is to be drawn tangent to the directrix, of a radius R.

1. Lay off on AB, a distance equal to $2\pi R$, and divide the circle and AB into any convenient number of equal parts, 12, 16 or 24.

2. Draw the path of the center of the circle, parallel to AB, at a distance equal to R.

3. Erect perpendiculars to the line of centers at each of the divisions of AB. The intersections will be successive positions of the center of the generating circle.

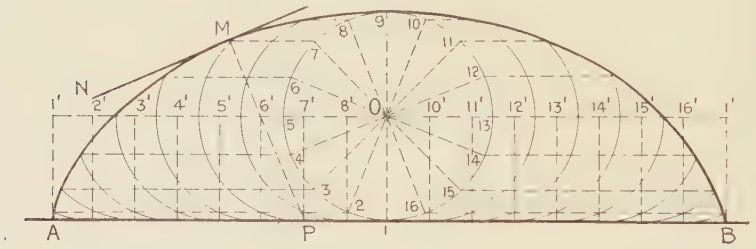


Fig. 77.—The cycloid.

4. From the divisions, 1, 2, 3, etc., of the generating circle, run projectors parallel to AB.

5. From each of the successive positions of the center, 1′, 2′, 3′, etc., describe arcs intersecting the corresponding projectors.

6. Through these intersections draw a smooth curve, which will be the required Cycloid.

142. Problem 54. To draw the tangent to the Cycloid at any point on the curve.

Construction.—Let M (Fig. 77) be the given point, and P be the point of tangency with AB of the generating circle, when M is its intersection with the curve.

1. Draw MP. This line is the normal at M.

2. Draw the tangent perpendicular to MP.

Note.—It is owing to the fact that the normal to the cycloid at any point passes through the point P, where the generating circle and directrix are tangent, that the cycloid is a correct curve for use in gear outlines. It is this property that makes it possible to maintain a constant angular velocity ratio in gears, as will be seen when the study of Kinematics is taken up.

143. Problem 55. To draw an Epicycloid and a Hypocycloid. The only difference between drawing these curves and the cycloid, is that the directrix now is a circle instead of a straight line. Fig. 78 shows both curves. AB is an arc of the pitch circle, as the directrix is usually called in Machine Design, and the center of the generating circle of the Epicycloid is at C, and its path is the arc CD (center at O). The center of the Hypocycloidal circle is at E, and its path is the arc EF, center at O.

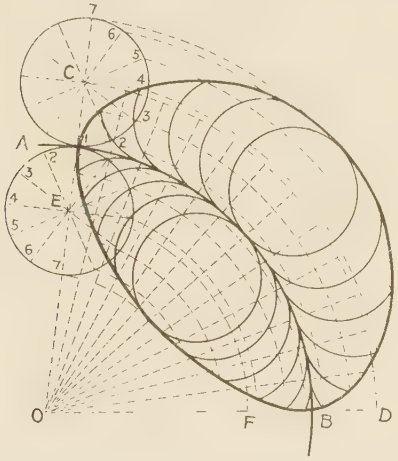


FIG. 78.—The epicycloid and hypocycloid.

Construction.—1. Divide the generating circles into any convenient number of equal parts.

2. Step off these arcs on AB (Art. 125).

3. From the points 1, 2, 3, etc., draw arcs with their centers at O.

4. Through the divisions on AB draw radial lines intersecting CD and EF in the points occupied by the centers as the circles roll on AB.

5. From these center points strike arcs with radii equal to those of the generating circles, intersecting the arcs of operation (3).

6. Through the respective intersections draw the curves.

The tangents are drawn precisely as the tangents to the cycloid are drawn; that is, the normal at any point passes through the point of tangency of the generating circle and the pitch circle.

144. Problem 56.—To draw an Involute to a Circle.

Let AB (Fig. 59) be the directing circle, or **Evolute,** with its center at O.

Construction.—1. Divide any arc of AB, as, for example, the semicircle, into any number of equal parts.

2. At the points 1, 2, 3, etc., draw tangents in the direction of the origin, A.

3. Rectify each arc from the origin and lay off the distance on the tangent.

4. Through the extremities of the tangents draw the curve.

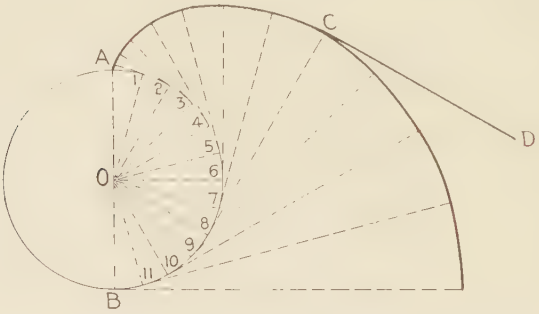


FIG. 79.—The involute.

Note 1.—This curve is that described by the end of a taut cord as it is unwound from a post. It is not a closed curve, but a spiral that keeps growing in expanse with every revolution.

Note 2.—An involute can be described in the same way, using any curve or polygon as the evolute.

Note 3.—The **Volute,** used in architecture, is a spiral, the involute of an enlarging square.

145. Problem 57. To draw the Tangent to an Involute at any point on the curve.

The tangents to the evolute are all normals to the involute (Art. 129), hence all that is necessary is to draw the perpendicular to the normal at the given point. Thus at the point C (Fig. 79) draw CD perpendicular to C8, the normal.

Note.—The fact that the normal to the involute is tangent to the evolute makes the curve useful in gearing in the same way as in the case of cycloidal curves.

146. Problem 58. To draw the projections of a Helix, having given its Pitch and Diameter.

Let OP (Fig. 80) be the axis, o'12 the pitch, and D the diameter.

Construction.—1. Draw the circle and divide it into any number of equal parts.

2. Divide the pitch into the same number of equal parts.

3. Draw perpendiculars to the axis from each of the divisions made in operation (2).

4. Draw projectors from each of the divisions of the circle, intersecting the corresponding perpendiculars from the axis.

5. Draw the curve through these intersections.

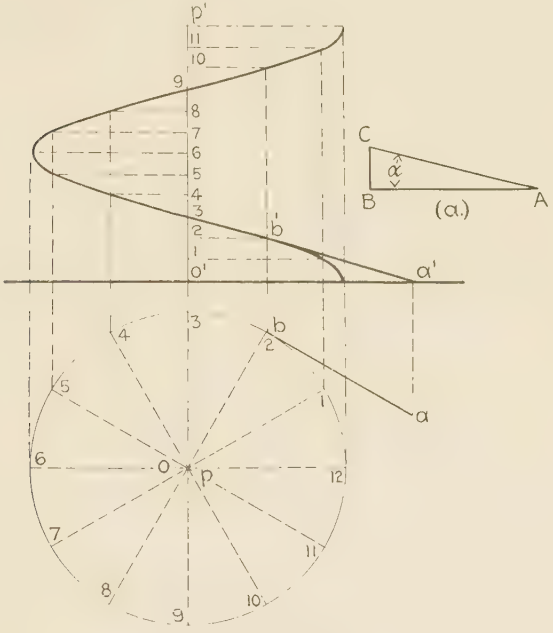


Fig. 80.—The helix.

147. The Angle of the Helix.

From the definition (Art. 128), we see that the generating point on the helix ascends a given distance in traversing a given arc, and that the rate of ascent is constant. This means that any element of the curve makes a certain angle with the horizontal (the axis of the helix being taken in all these problems as vertical), and that all the elements in any one helix make the **same angle** with the horizontal. Thus, in Fig. 80 (a), if AB is the length of an elemental arc of the circle, and BC is the length of an elemental division of the pitch, the hypothenuse of the right triangle ABC will yield the inclination and the length of an element of the

helix. Therefore, if a right triangle, having its base angle equal to α, be wrapped about a cylinder, the hypothenuse will become the helix. From this it is evident that tan. $\alpha = \frac{\text{pitch}}{\pi \text{ Diam.}}$

148. The Tangent to the Helix.

If the right triangle, just mentioned, be unwrapped from the cylinder, its hypothenuse becomes the tangent at the point where the unwrapping ceases.

This is so because the tangent must make the same angle with the horizontal that the helix does, because it contains two consecutive points (**one element**) of the curve.

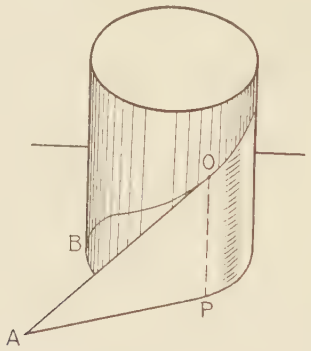


FIG. 81.—The generation of a helix.

Therefore, the tangent becomes the helix if it is rolled on the cylinder, and the length of the helix from its origin to the point of tangency is equal to the length of the tangent from the plane of the origin to the point of tangency. Referring to the picture, Fig. 81, will show that AO = BO, and further that AP = BP. AP is the *H*-projection of the tangent, and BP is the *H*-projection of the helix. From this the rule is derived: **The *H*-projection of the Helix is of the same length as the *H*-projection of the Tangent from the plane of the origin to the point of tangency.**

149. Problem 59. To draw the Tangent to the Helix at any point on the curve.

Construction.—Let B (Fig. 80) be the given point. 1. Draw the tangent to the circle at b.

2. Rectify the arc b12, and lay it off on the tangent ab.

3. Project a to a′ on *GL*.

4. a′b′ will be the *V*-projection, and ab the *H*-projection of the tangent required.

150. EXERCISES IN CURVED LINES

Note.—No ground line is necessary in drawing any of the curves but the helix.

518. Draw an Ellipse, AB = 5 in., FF′ = 3 in. Draw a tangent at a point on the curve 2 in. from F. Draw two tangents from a point 4 in. from F and 2 in. from F′.
519. Draw an Ellipse, AB = 4½ in., and CD = 3 in. Locate F and F′, and draw a tangent at a point on the curve 1 in. from A. Draw two tangents from E, which is 3 in. from F′ and 2 in. from F.
520. A and B are two tangent ellipses of equal size. The foci of A are C(1, 0−2)D(3½, 0−2), and the more distant focus of B is at E (5, 0+1). Draw the two ellipses.
521. Draw a Hyperbola. FF′ = 3 in. and the "constant difference" = 2 in. Draw the tangent at a point on the curve 2 in. from F. Draw a tangent from a point 2 in. from F′ and 2½ in. from F.
522. Draw a Parabola with the distance between focus and directrix 1 in. Draw tangents from assumed points on and outside the curve.
523. Draw a Parabola whose distance from focus to vertex is ¾ in. Draw tangents from assumed points on and outside the curve.
524. Draw a Cycloid with a 1½-in. generating circle. Draw a tangent at the point on the curve at $\frac{5}{16}$ of the revolution.
525. Draw an Epicycloid and a Hypocycloid on a pitch circle of 6-in. radius, with 1¼-in. generating circles. Draw the normals to each curve at $\frac{5}{16}$ of a revolution from the origin.
526. Draw an Involute of a 1-in. circle.
527. Draw an Involute of a ½-in. square.
528. Draw an Involute of a hexagon of ½-in. sides.
529. Draw an Involute of a ¾-in. equilateral triangle.
530. Draw a Helix of 2-in. diameter, 3-in. pitch. Draw its tangent at 60° from the origin. Also one at 90°.

SURFACES

151. Definitions.

A line, straight or curved, moving in accordance with some law, generates a **Surface.** If the generating line be straight, and moves along two parallel or intersecting straight lines, the surface generated will be a **Plane.** There is but one kind of plane, all other kinds of surfaces are curved.

152. Curved Surfaces are divided into three classes, **Single-Curved, Double-Curved and Warped** surfaces.

A **Single-Curved Surface** is generated by a straight line following a curved director in such a way that any two consecutive positions of the line shall lie in a plane.

A **Warped Surface** is generated by a straight line moving so that no two of its consecutive positions shall be in the same plane.

Note.—Single-curved and warped surfaces are sometimes called **Ruled** surfaces, because they are made up of straight lines. Through every point of a ruled surface at least one straight line can be drawn in the surface.

A **Double-Curved Surface** is generated by a curve moving along a curved line director.

Note.—There can be no straight lines in any double-curved surface.

Generatrix.—The line generating a surface.

Element.—Any position of the generatrix.

Consecutive Elements.—Two successive positions of the generatrix, having no assignable distance between them.

A **Surface of Revolution** is a surface generated by the revolution of any line, straight or curved, about a straight line axis. Surfaces of revolution may be single-curved, double-curved, or warped.

153. Classification of Surfaces.

Single-Curved Surfaces are of three varieties:

Cones, all the elements intersect in a point.

Cylinders, all the elements are parallel.

Convolutes, consecutive elements intersect **two and two,** no three elements having a common point.

Double Curved Surfaces are mostly surfaces of revolution, spheres, ellipsoids, paraboloids, tori, and combination surfaces. They are sometimes convex, sometimes concave, and sometimes both. The annular torus has both in continuous surface.

Warped Surfaces are numerous in their variety, but the most important are the helicoid and hyperboloid of revolution of one nappe.

154. The following table will give a convenient summary of the surfaces.

Single-Curved	Cone—all elements intersecting	
	Cylinder—all elements parallel	
	Convolute—consecutive elements intersecting.	
Warped	Helicoid	Ruled Surfaces.
	Hyperboloid of Revolution	
	Hyperbolic Paraboloid	
	Conoid	
	Cylindroid	
	Warped Cone	
	Cow's Horn.	

Double-Curved	Sphere
	Oblate Spheroid
	Prolate Spheroid
	Paraboloid (both varieties)
	Hyperboloid of Revolution of Two Nappes
	Torus.
Surfaces of Revolution	Right Circular Cone, Right Circular Cylinder
	Hyperboloid of Revolution of One Nappe
	Sphere, Spheroids, Ovoids, etc.

155. Tangent Planes and Developments.

The most important problems in surfaces are to be found in drawing tangent planes to surfaces, the intersections of surfaces, and developing the surfaces.

A **Tangent Plane** to a surface at any point is a plane composed of all the tangent lines to all the lines of a surface passing through that point. As two intersecting lines determine a plane, the general rule for tangent planes is this: **Any two tangent lines at any point on any surface determine the tangent plane at that point.**

Any plane, passed through the point of tangency so as to intersect a surface and its tangent plane, will cut from them a line and its tangent.

In ruled surfaces, one and sometimes two straight lines of the surface can be drawn through each point in the surface, therefore this line (element, as it always is) can be used for one of the tangents in the foregoing rule. This is true, because a straight line is its own tangent. On this account, the rule will read: **"In ruled surfaces, one tangent and the element at any point will determine the tangent plane at that point."**

Normal.—The normal line at any point of a curved surface is the perpendicular to the tangent plane at that point. The normal plane is any plane containing the normal line.

Development.—By **Development** is meant the rolling out of a surface into a plane figure, which can be rolled, or folded, back to make the original surface. Only single-curved surfaces are strictly developable, because they only have consecutive elements lying in planes. **Approximate developments** can be made of other surfaces, as will appear when the surfaces are taken up separately.

PROBLEMS RELATING TO SINGLE-CURVED SURFACES

156. The Cone.

Definitions.—A **Cone** is generated by a straight line moving so as to constantly touch a curved line directrix, while one point in the generatrix remains stationary.

Apex or **Vertex,** the stationary point on the generatrix.

Base, a limiting plane that cuts all the elements.

Axis, a straight line through the apex and the center of the base.

A **Frustum** is a portion of the surface between two parallel planes on the same side of the apex, which cut all the elements.

157. Discussion.—The surface of the cone that lies between the base and apex is usually all that is considered. Theoretically, however, the surface is unlimited, and extends beyond both limits to infinity. This theoretical extension creates two divisions of the surfaces, one on each side of the apex, called respectively the **Upper Nappe** and **Lower Nappe.** In practical work this condition almost never exists, and is usually ignored.

Varieties of Cones.—The different varieties of cones are distinguished by their bases and axes. The bases may be circular, elliptical, hyperbolic, etc., or combinations of curves, and the axes are either perpendicular (**right**) or oblique to the bases. Thus we designate a cone as a right circular cone, an oblique elliptical cone, etc. The base and apex, or the base and axis, are sufficient to determine the cone.

158. Limits.—A cone, whose apex is at infinity, becomes a **cylinder,** and one, whose apex is in the plane of the base, becomes a **plane.**

159. A Tangent Plane to a cone at any point on the surface is tangent all along the element through that point. This element is called the **Element of Contact.** As all elements pass through the apex, **every tangent plane must contain the apex.** This fact is of great importance in passing tangent planes. It means that **any line through the apex,** except lines inside the cone, **will lie in one or two tangent planes.** From this fact, and from Art. 155, we derive the special rule for passing tangent planes to cones.

Rule. One tangent line and the apex are sufficient to determine a tangent plane to any cone.

Axiom.—If the base of a cone is in H, the H-trace of any tangent plane will be tangent to the base.

160. Development of Cones.—When the surface of a cone is rolled out, we find that there is one point common to all elements, the apex, and this point will remain a point in the development. From this point all the elements radiate in fan shape, making up an integral of all the minute triangles that form the surface of the cone. In the case of a right circular cone, the elements are all of equal length, and the base rolls out into an arc of a circle, whose radius is the length of the elements (sometimes called **"slant height"**). The length of the arc thus derived is equal to the circumference of the base. All other cones have elements of varying length, and their bases will not roll out in circular paths. The development of all cones, except the right circular, is accomplished by **"triangulation"**; that is, dividing the surface into small triangles, and laying them out flat in order and (approximately) true size. An exception to this may be found in an oblique elliptical cone that may have a circular right section. If such be found, the circular method is better than triangulation.

PROBLEMS RELATING TO CONES

161. Problem 60. —To Locate a Point on the Surface of a Cone.

Construction.—See Fig. 82.

1. Assume either projection of the point O, say the H-projection.
2. Through o and b(the apex) draw the H-projection of the element.
3. Project c to c′ on GL.
4. Draw b′c′.
5. Project o to o′ on b′c′.

162. Problem 61.—To find the Line of Intersection of a Cone and Plane.

Analysis.—Find the piercing points of a reasonable number of elements with the given cone. The line through these points will be the line of intersection.

Construction.—Shown in Fig. 83.

Note.—In case the piercing points are difficult to locate by the ordinary process, the best and easiest way to accomplish it is by the method of auxiliary plane projection described in Art. 105.

In case the plane is parallel to *GL* the operation is easiest by the use of the profile.

163. Problem 62.—To find the Piercing points of a Line and a Cone.

Analysis.—1. Pass a plane through the given line and the apex.

2. Obtain the elements cut by this plane.

3. Where the elements intersect the given line will be the required piercing points.

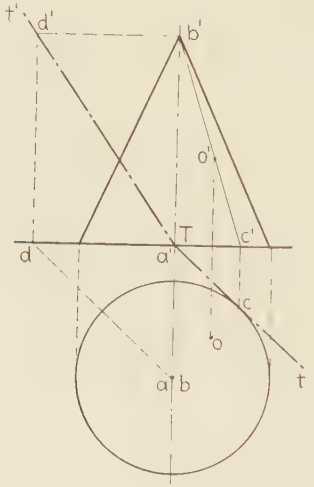


Fig. 82.—A point on the surface of a cone, and the tangent plane at that point.

Construction.—Let the cone PO (Fig. 84) and the line AB be given.

1. Pass a plane through AB and P, find its *H*-trace, cd. (*V*-trace not necessary.)

2. Find the points, e and f, where cd intersects the base.

3. Draw the elements, pe and pf.

4. Find the intersections, m and n, of the elements and ab.

5. Project m and n to m′ and n′ on a′b′. M and N will be the required piercing points.

Which operation determines whether AB will pierce the cone or be altogether outside of it?

Note.—There are occasions when a plane cutting a number of elements can be used to better advantage than a plane through the apex. Such a plane will cut a curve from the cone surface, and the piercing points will be at the intersections of the given line with the curve.

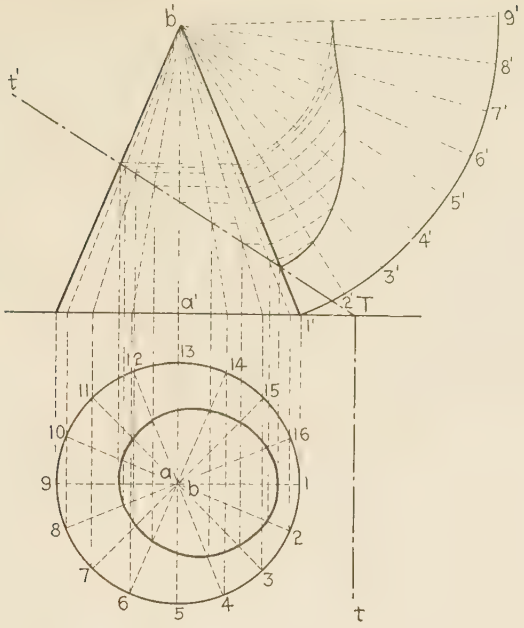


FIG. 83.—A cone intersected by a plane.

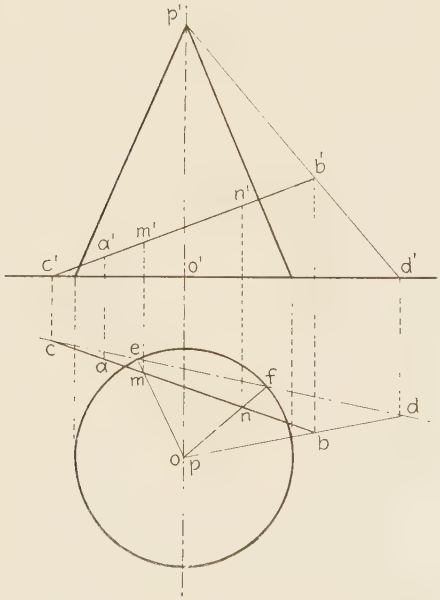


FIG. 84.—Piercing points of a line and a cone.

164. If the base of the cone is not in H, V or P, the problem is more complicated, but the method is unchanged. The student must accustom himself to problems that involve new applications and changed conditions of the processes illustrated.

165. Problem 63.—To pass a plane Tangent to a Cone at a point on the surface.

Analysis.—Read the rule given in Art. 159.

Construction.—Refer to Fig. 82, Art. 162. Let O be the given point.

1. Draw the element through O. This is the **Element of Contact,** and one line of the plane.

2. Where the element of contact intersects the base, draw the tangent to the base.

This is the H-trace of the required plane, and the V-trace may be obtained in several ways known to the student.

166. Problem 64. To pass a plane Tangent to a Cone through a Point outside the surface.

Analysis.—1. Draw a line through the apex and the given point. (See Art. 159.)

2. Find the piercing point of this line with the plane of the base.

3. From this point draw one of the two possible tangents to the base.

4. Pass the required plane through the tangent line and the apex.

Construction.—Let the student make it.

Note 1.—If the line through the given point and the apex is parallel to the base, the tangent line in the plane of the base will be parallel to said line, and may be drawn on either side of the base.

Note 2.—If the line through the apex is too nearly parallel to the plane of the base to obtain the piercing point within the limits of the drawing, a plane may be passed parallel to the base, nearer the apex, and a tangent may be drawn to the new base.

167. Problem 65. To pass a plane Tangent to a Cone parallel to a given line.

Analysis.—1. Draw a line through the apex parallel to the given line.

2. Having this line, finish as in Problem 64.

Construction.—Let the student make it.

Impossible Conditions.—When the line through the apex parallel to the given line is inside the cone, the problem is impos-

sible. In such a case the line will pierce the plane of the base inside the base. Otherwise the problem is possible.

168. Problem 66.—To pass a plane Tangent to a Cone, making a given angle with *H*.

Limitations.—Only certain conditions will allow a solution of this problem.

1. If the cone be a right circular cone, with its base on *H*, there is only one angle possible, and that is the anlge that its elements make with *H*.

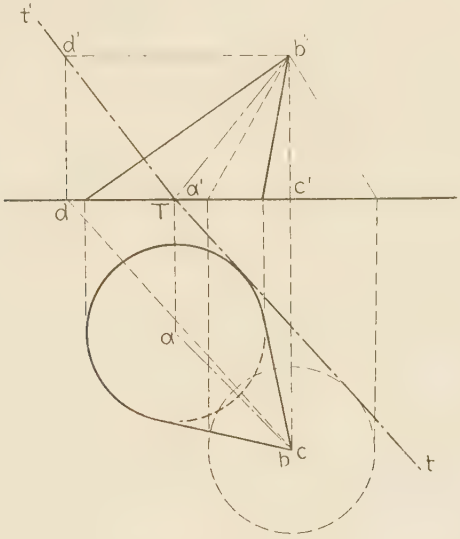


Fig 85.—Plane tangent to a cone, and inclined 60° to the horizontal.

2. If the base be in some plane not parallel to *H*, there are many angles at which tangent planes may be passed, though not all.

3. If the base be in *H*, and the axis oblique, different angles, but not all, may be assigned to the problem.

4. If the base be in *H*, and a different curve from the circle, even though the axis be perpendicular, there will be some variety to the angles that may be required.

Each problem will fix its own limitations.

Construction.—Let it be required to pass a plane tangent to the cone AB (Fig. 85) making 60° with *H*.

1. With B as its apex, draw the projections of a 60° cone, axis BC.

2. Draw the H-trace of T, tangent to the bases of the two cones. This may be drawn on either side of the cones.

3. Locate the V-trace of T by any lines through B, lying in T.

Note.—Under certain conditions four planes can be passed making the same angle with H, all tangent to the cone.

169. Problem 67. To develop the surface of a right circular cone.

Analysis.—Read Art. 160.

Construction.—Let it be required to develop the truncated cone in Fig. 83; that is, the surface included between the base and the plane T.

1. Divide the base into any convenient number of equal parts, and draw the elements.

2. From b′, the apex, on the V-projection, describe an arc of a radius equal to the true length of the elements.

3. Make this arc equal in length to the circumference of the base of the cone, and divide it into the same number of equal parts. (See Note below.)

4. Measure the true length of the truncated elements, and lay them off on the corresponding radii of the **Pattern** (development).

5. Draw a smooth curve through these points.

Note.—The accurate way to lay off this arc equal in length to the circumference is by this equation: $\frac{R}{r} = \frac{360°}{\theta°}$, or $\theta° = \frac{r}{R} 360°$, where R equals the slant height of the cone, r the radius of the base, and θ is the number of degrees in the angle of the development. It can be done more roughly by stepping off small arcs with the dividers.

170. Problem 68. To develop the surface of an oblique cone.

This is accomplished by triangulation, and it is as nearly exact as the chord of a small arc is to the length of the arc.

Construction.—The cone PO (Fig. 86).

1. Divide the base into a number of equal arcs.

2. Draw the elements of these divisions, dividing the cone into small (approximate) triangles.

3. Measure the true length of all the elements.

Note.—By a little care in the division this can be done in pairs.

4. Rectify one of the equal arcs.

5. Beginning (preferably) with the longest element, O1, construct triangles equal to the (approximate) true size of those in the cone, as shown in the development at the right.

Second Method.—A method often used in the development of oblique cones in the "Sphere Method." The process is as follows:

1. Draw the projections of a sphere, of any convenient size, with its center at the apex.

2. Obtain the piercing points of the various elements on the surface of the sphere.

3. The segments of the elements thus obtained are all of equal length, and will therefore be developed on the arc of a circle, like the right circular cone.

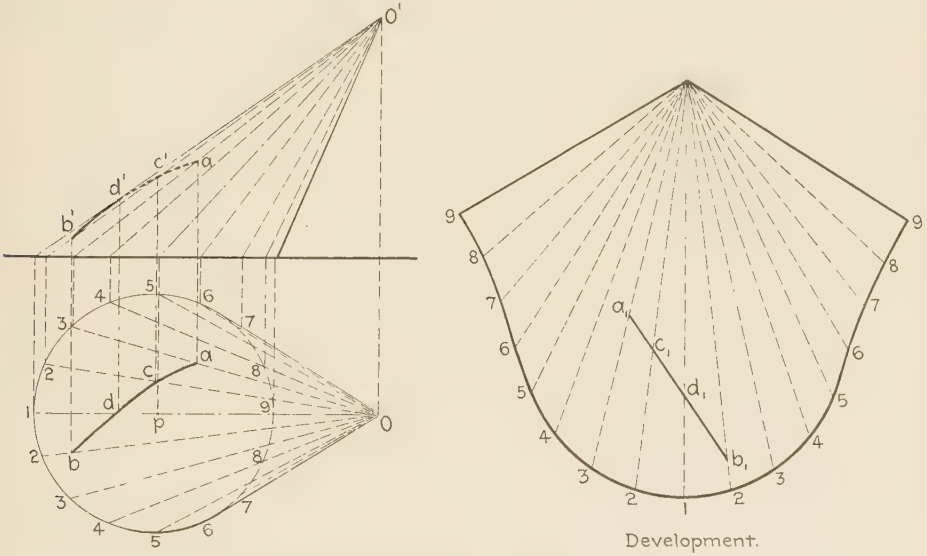


FIG. 86.—The shortest line between two points on the surface of a cone.

4. The remainder of each element can then be measured, and laid off on the development in its proper place.

Let the student make the construction.

171. The Problem of the Shortest Path.

The shortest line that can be drawn on any surface from one point to another is a straight line on the development. Let A and B (Fig. 86) be two points on the surface of the cone.

1. Locate them on the development, see a_1 and b_1.

2. Draw a straight line from a_1 to b_1, and locate its intersections, c_1, d_1, etc., with the intervening elements.

3. Return all these points to their original positions in the projections of the cone, and draw the curve through them.

Note.—The shortest path on a cylinder is always a helix, a circle, or a straight line.

172. EXERCISES

531. Find the points in which A(1+2−½)B(4+½−3) pierces the cone, whose axis is C(1½, 0−2½) D(3½+3−1¾), and base a 2 in. circle in *H*.
532. Draw the projections of a right circular cone, tangent to *H*, its axis A(2, 0−1) B(4+1−2¼). Find where it is pierced by C(2+2−2½) D(4, 0−1¼).
533. Draw the projections of a right circular cone, axis A(2+1−1½) B(4+1−1½), base a 1¾-in. circle at B. Find where it is pierced by C(2+½−½) D(4+2−2).
534. Draw the shortest line that can be drawn on the cone in Ex. 533 from E(3+¾, x) on the rear surface to F(3½+1¼, y) on the front surface.
535. Draw the shortest line from M(2½+¼, x) on the rear surface to N (3½+1, y) on the front surface of the cone in Ex. 532.
536. Find the intersection of the plane T(1+2) 5(3−3) with the right circular cone of 2-in. base, whose axis is A(3, 0−1) B(3+2½−1). Find the true size of the section.
537. Develop the cone in Ex. 536 between *H* and T.
538. Given the cone in Ex. 536. Intersect it by four planes, all perpendicular to *V*, that will cut it respectively in a circle, ellipse, hyperbola and parabola. Show each curve in its true size.
539. Let A(1½+2½−2) B(4+1−1) be the axis of a right circular cone, of 2-in. base, apex at A. Draw its projections. Find the intersection with the plane T(4+3) 1(4−3).

Note.—This can be much easiest worked by the auxiliary plane method described in Art. 105.

540. Develop the surface of the cone in Ex. 539 between the apex and T.
541. M(2+½−2) N(3+2−1½) is the axis of a right circular cone of 1½-in. diameter at M. Find its intersection with *H*.
542. Develop the surface of the cone in Ex. 541 between the apex and *H*.
543. A(3+1−½) B(3+1−2½) is the axis of a right circular cone, base a 2-in. circle at A. Draw the traces of a plane tangent at O(2½+¾, x) on the surface.
544. Draw the traces of a plane, tangent to the cone in Ex. 543, containing M(2+1½−1½).
545. Draw the traces of a plane, tangent to the cone in Ex. 543, parallel to A(1+1−1) B(2+2−2½).
546. Draw the traces of a plane, making 60° with *H*., tangent to the cone in Ex. 543,
547. Draw the traces of the plane, tangent to the cone in Ex. 532, containing the point C(3+¾, x) on the rear surface.
548. Draw the traces of a plane, tangent to the cone in Ex. 533, containing the point N(2+½−1).

549. Draw the traces of a plane, tangent to the cone in Ex. 533, parallel to O(1+2−3) P(2+1−1½).
550. Draw the traces of a plane, tangent to the cone in Ex. 533, making 45° with *H*.
551. Draw a right circular cone, axis A(3+¾−¾) B(3+2−1½), base a 1½-in. circle at A. Draw the traces of a plane tangent at O(2¾+1, x) on the surface.
552. Draw the traces of a plane, tangent to the cone in Ex. 551, containing the point P(2+1−1½).
553. Draw the traces of a plane, tangent to the cone in Ex. 551, parallel to A(1+2−3) B(2+1−1½).
554. Draw the traces of a plane, tangent to the cone in Ex. 539, containing K(3+½−½).
555. Draw the traces of a plane, tangent to the cone in Ex. 539, containing E(3+1, x) on the rear surface.
556. Draw the traces of the plane, tangent to the cone in Ex. 532, containing the point C(3+¾, x) on the rear surface.
557. Draw the traces of a plane, tangent to the cone in Ex. 532, containing the point F(4+½−1).
558. Given an oblique cone, axis A(2½, 0−2) B(4+2−1), base a 2-in. circle in *H*. Draw the traces of the plane tangent at M(2, x−2½) on the surface.
559. Draw the traces of a plane, tangent to the cone in Ex. 558, containing N(3½+½−2).
560. Draw the traces of a plane, tangent to the cone in Ex. 558, parallel to X(2+3, 0) Y(1+1−3).
561. Draw the traces of a plane inclined 75° to *H*, tangent to the cone in Ex. 558. How many such planes are possible?
562. Draw a right circular cone tangent to *H*, axis M(2+1−1½) N(4, 0−1½). Draw the traces of a tangent plane through O(3+⅝, x) on the rear surface.
563. Draw the traces of a plane, tangent to the cone in Ex. 562, containing P(3+2−½).
564. Given a right circular cone, axis A(2, 0−1½) B(2+2½−1½), base 2-in. diameter at A. Place a 1¼-in. sphere so as to be tangent to *H*, *V* and the cone. Draw the traces of a plane tangent to the sphere and cone, but not between them.
565. Draw the traces of a plane tangent to two right circular cones, both of whose apices are in the point M(3+1−1½), one having a 2-in. base in *V*, and the other a 1-in. base in *H*.

173. PRACTICAL EXERCISES

566. Draw the patterns (development) for the funnel (Fig. 87) to a convenient scale.
567. Draw the patterns for the ventilator hood and deflector (Fig. 87) to scale. Use two spaces.
568. Draw the patterns for the conical offset pipe (Fig. 87) to any convenient scale.

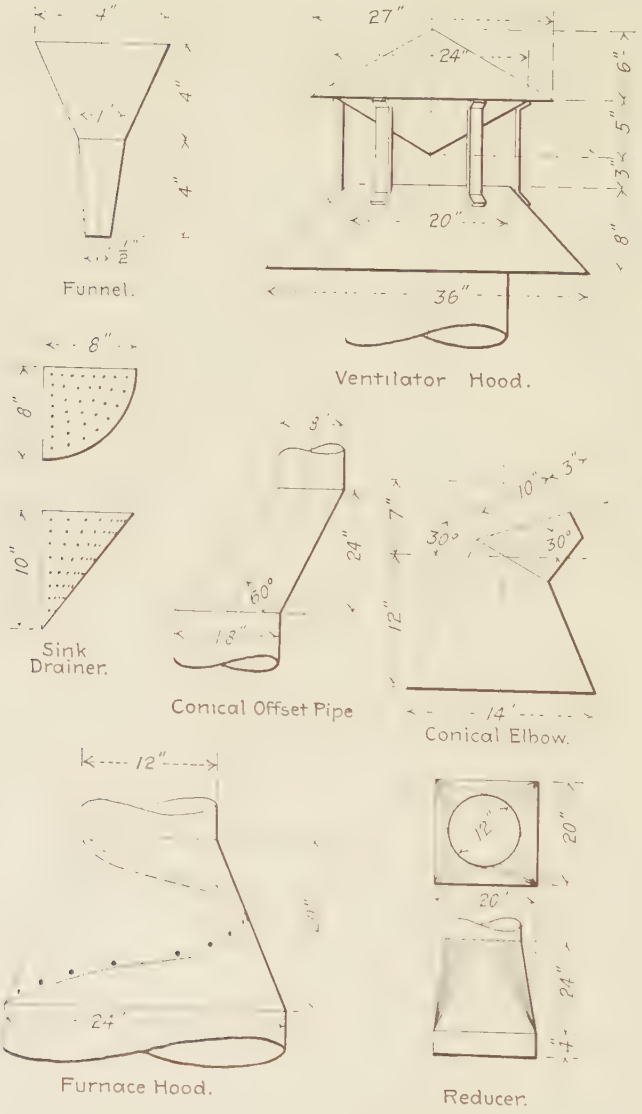


FIG. 87.—Practical examples of conical surfaces.

569. Lay out the patterns for the sink drainer (Fig. 87) to any convenient scale.

570. The conical elbow, shown in Fig. 87, is composed of two cones, the right sections of the larger being circles. The smaller has an elliptical lower base, and the upper base to be determined. Draw both projections, find out the true shape of the spout, and lay out the patterns to scale. Use two spaces.

571. Develop the conical furnace hood (Fig. 87) along a "spiral seam." Make the plan of the conical helix a Spiral of Archimedes, making one complete revolution. Omit lap and rivet holes. Any convenient scale.

572. Lay out the patterns for the chimney reducer, shown in Fig. 87, to any convenient scale.

THE CYLINDER

174. Definition.—A cylinder is a single-curved surface generated by a line moving along any curved-line directrix, always remaining parallel to a given line. It is a **cone with its apex at infinity.**

The **Surface** of a cylinder is usually taken to be that portion which is included between two plane bases. The bases may be parallel or not. Theoretically the cylinder is infinite in extent.

Axis.—The axis is a line through the center of a right section, parallel to all the elements. It is a useful line in defining the cylinder in space.

Varieties of Cylinders.—The characteristic distinction between cylinders is the curve of their **right section.** This is at some variance to the cone distinctions, which are founded on base and axial peculiarities. The reason for this is that it is absolutely necessary to know the right section of a cylinder in most of the problems arising; as, for instance, in its development, the right section always rolls out in a straight line. Also, in most tangent problems, the right section has to be considered. So, when a cylinder is spoken of as circular, elliptical, etc., **a right section is understood.**

Limits.—When the directrix becomes a straight line, the cylinder becomes a plane, and when the directrix becomes a point the cylinder becomes a line.

175. Tangent Planes to Cylinders.—A plane which contains two consecutive elements is tangent. **The Tangent Plane is parallel to the axis, and to every element.** The two consecutive elements contained in the tangent plane form the **Element of Contact.** This element of contact is one of the two tangent lines

necessary to determine the tangent plane. From this fact we derive the special rule for passing planes tangent to cylinders.

Rule.—A tangent line and a line parallel to the axis, intersecting the tangent, determine a tangent plane to a cylinder.

Axiom.—When the base of a cylinder is in H, the H-trace of any plane, tangent to the cylinder, will be tangent to the base curve of the cylinder.

Axiom.—A plane cannot in general be tangent to two cylinders unless the axes are parallel. The exceptions, which are not frequent, will only be found when two outside elements of the two cylinders are in the same plane. That plane will be the tangent plane. If two cylinders have parallel axes, four planes tangent to both can usually be passed. Two cylinders of equal diameter, with axes intersecting, can have two common tangent planes outside.

176. Development.—As was before mentioned, the right section of a cylinder rolls out into a straight line. This is true, because the right section is perpendicular to all the elements, and when they are rolled out they remain parallel, and the plane perpendicular to them cuts their plane in a straight line perpendicular to them. Therefore, the first thing to do is to **rectify a right section.** Then lay off perpendiculars, properly spaced, with the true lengths of the various elements laid off in order.

Axiom.—The cylinder included between two right sections becomes a rectangle in development.

PROBLEMS RELATING TO CYLINDERS

177. Problem 69.—To locate a point on the surface.

Analysis.—1. Assume one projection of the point.

2. Draw a line through that projection parallel to the axis. (This will be the element on which it lies.)

3. Find the intersection of this element with the base.

4. Project this intersection to the other projection of the base and draw the other projection of the element.

5. Project the point to the second projection of the element.

Let the student make the construction.

178. Problem 70.—To find the intersection of a plane and a cylinder.

Analysis.—1. Draw a sufficient number of elements.

2. Obtain the piercing points of these elements with the plane.

3. Draw a smooth curve through these piercing points in order.

179. Problem 71.—To find the points in which a straight line pierces a cylinder.

There are two methods: 1. to use when the base of the cylinder lies in one of the planes of projection; 2. to use when the base is in space.

Analysis. First Method.—When the base is in H, V or P.

1. Pass a plane through the line parallel to the axis.

2. At the points in which the trace of this plane crosses the base, draw elements of the cylinder.

3. The points in which these elements are intersected by the given line will be the required points.

Second Method.—When the base is in space.

1. Draw a sufficient number of elements.

2. Pass a plane through the line, preferably a projecting plane.

3. Find the intersections of this plane with enough of the elements to yield a curve that will be intersected twice by the given line.

4. The required points will then be where the given line crosses the curve.

180. Problem 72.—To pass a Plane Tangent to a Cylinder through a Given Point on the Surface.

Analysis.—1. Through the given point draw a line parallel to the axis.

(This will be one line of the required plane, the element of contact.)

2. At the point where the element of contact crosses any curve of the surface (preferably one of the bases), draw the tangent to that curve.

3. Pass the plane through the element and tangent.

Construction—Let O (Fig. 88) be the given point on the surface of the elliptical cylinder whose axis is AB.

1. Draw the element OP.

2. Draw the tangent to the base at P.

3. This tangent will be the H-trace of the tangent plane.

4. Draw the V-trace by any of the known means.

Note.—This construction, as illustrated in Fig. 88, is only possible when the base of the cylinder is in H, V or P. If the base is in an oblique plane, the tangent will not be one of the traces of the plane. The **analysis** will apply to **any** situation of the cylinder.

181. Problem 73. To pass a Plane Tangent to a Cylinder through a Point outside the surface.

Analysis.—1. Draw a line through the point parallel to the axis.

2. Find the point in which this line pierces the plane of the base.

3. From this piercing point draw a tangent to the base.

4. The parallel line and the tangent line will determine the required tangent plane.

Note.—Let the student assume an outside point and make his own construction for a cylinder like the one illustrated in Fig. 88. A more difficult construction is shown in Fig. 88*a*.

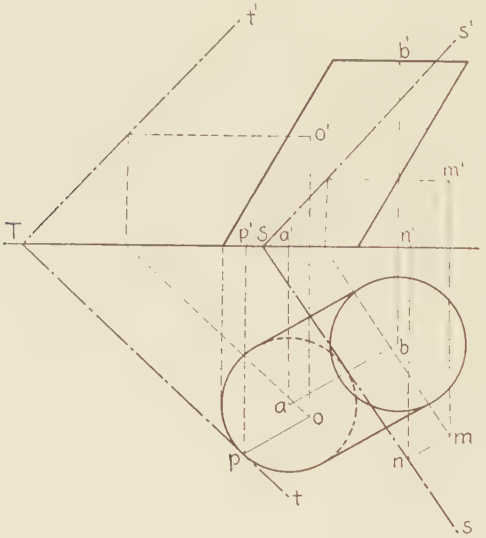


Fig. 88.

Construction.—Let the cylinder be a circular cylinder of $\frac{3}{4}$-in. diameter, whose axis is A B (Fig. 88*a*). Let it be required to pass a tangent plane through M, a point outside the surface.

1. Draw the line MN, parallel to A B through M.

2. Pass a plane Q through A, perpendicular to A B. (This will be the plane of the base, cutting a circle from the surface.)

3. Find the piercing point, N, of MN with Q.

4. Revolve A and N into H about Qq.

5. Draw a $\frac{3}{4}$-in. circle about a_1 as a center.

6. From n_1 draw a tangent to the circle.

7. Counter-revolve the tangent n'o into the plane Q.

8. Pass the plane S through NO and MN. It will be the required tangent plane.

Note.—Only the axis of this cylinder is drawn as the problem can be worked without drawing the projections of the entire cylinder. To show it would unnecessarily complicate the figure. The student should familiarize himself with this sort of problem, and will find examples of it in Exercises 603–614, 620–627.

182. Problem 74. To pass a Plane Tangent to a Cylinder and parallel to a given line.

Analysis.—1. Pass a plane parallel to the axis and the given line. Why?

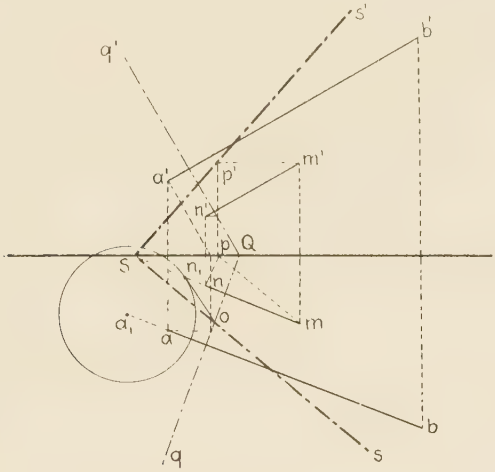


Fig. 88*a*.

2. Find the intersection of this plane and the plane of the base.

3. Draw a tangent to the base parallel to the intersection obtained in operation (2).

4. Pass the required tangent plane through this tangent line, parallel to the auxiliary plane, passed in operation (1).

Construction.—Let AB, Fig. 89, be the axis of the cylinder, and let it be required to pass the tangent plane parallel to XY.

1. Pass the auxiliary plane Q through AB parallel to XY.

2. Draw the required plane T, tangent to the cylinder and parallel to Q. The *H*-trace of T will be tangent to the base in *H*, and parallel to Qq, and the *V*-trace of T will be parallel to Qq′.

183.—If the cylinder be circular and its axis oblique, the best method for the foregoing problem is the **"Normal" method.** Its analysis is as follows:

1. Pass a plane through the axis parallel to the given line.

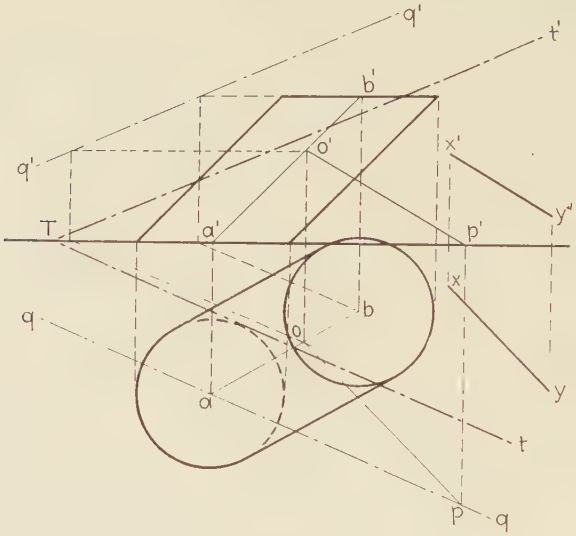


Fig. 89.

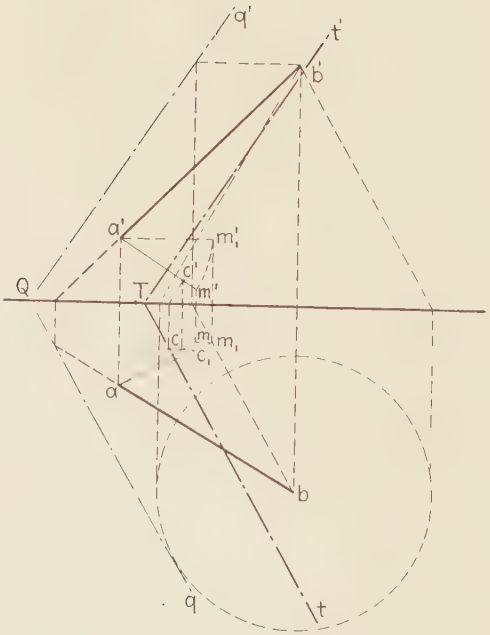


Fig. 90.

2. Pass a plane parallel to this plane at a distance equal to the radius of the cylinder (Problem 34).

The problem may be done in this way without drawing any elements or bases of the cylinder. Let the student prove that the "normal" method is correct.

184. Problem 75. To pass a Plane, inclined at any given angle with H or V, tangent to a Cylinder.

Limitations.—The limits of this angle must be between 90° and the angle between the elements and the stipulated plane.

Analysis.—1. Pass an auxiliary plane through the axis making the given angle with H or V, as required; see Problem 40.

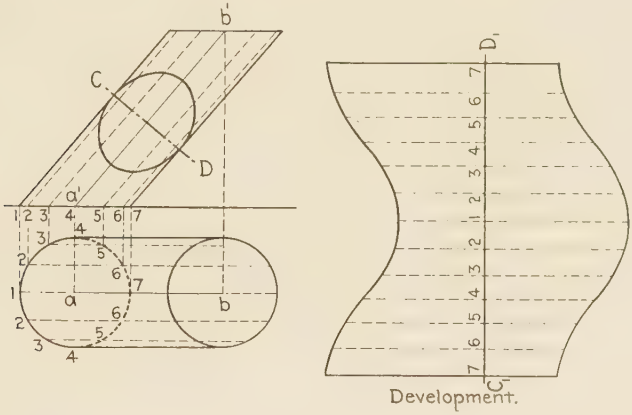


Fig. 91.

2. If the base of the cylinder is in H, draw the H-trace of the required plane, tangent to the base and parallel to the H-trace of the auxiliary plane.

3. Draw the V-trace of the required plane parallel to the V-trace of the auxiliary plane.

185. Normal Method.—When this problem is required of a circular cylinder, oblique to H and V, the normal method, like that in Art. 183, is the easiest and best.

Fig. 90 shows the construction of such a problem, in which AB is the axis and AC the radius of the given cylinder. The elements of the auxiliary cone make the given angle with H. T is one of the four possible required planes.

186. When the axis is parallel to H, and the angle between

the tangent plane and H is given, the problem is very easily solved by projecting on a plane perpendicular to the axis.

Note.—Four planes can be passed tangent to any cylinder, making the given angle, if the problem is possible.

187. Problem 76.—To Develop the Surface of a Cylinder.

Construction.—To develop the elliptical cylinder, whose axis is AB (Fig. 91).

1. Draw a sufficient number of elements.

2. Pass a plane perpendicular to the axis, as, for example, along the line CD.

3. Revolve this right section to show its true size.

4. Lay off the base line, C_1D_1, equal in length to the length of the ellipse, with divisions equal to the respective arcs, as shown by the corresponding numerals.

5. Through all these divisions draw perpendiculars to C_1D_1.

6. Lay off on these perpendiculars the true lengths of their respective elements, measured above and below the right section.

7. Draw a smooth curve through the extremities.

Note.—It is not always necessary to show both projections of the right section. In this case the H-projection is unnecessary.

EXERCISES

188. Where elements are needed, use at least 16 in these exercises.

573. Find the line of intersection, and its true size, of the plane T(1+3) 5(1−3) and the 2-in. circular cylinder, whose axis is A(2, 0−1) B(2+3 1).

574. Find the line of intersection, and its true size, of the plane S(+2)∞ (−3) with the cylinder in Exercise 573.

575. Find the intersection, and its true size, of the plane W(1+2) 5(1−3) with the elliptical cylinder, axis C(1, 0−1) D(3+3−2), base a 2-in. circle in H.

576. Find the intersection, and its true size, of the plane X(1+2) 3(3−3) and the cylinder in Exercise 575.

577. Find the true size of a right section of the cylinder in Exercise 575.

578. Find the intersection, and its true size, of the plane T(3+3) 5(3−3) with the $1\frac{1}{2}$ in. circular cylinder, whose axis is M(1+1−1) N(4+3−1).

579. Find the intersection, and its true size, of the plane P(3+3) 3(3−3) and the cylinder in Exercise 575.

580. Find the intersection, and its true size, of the plane S(4+3) 2(2−3) and the circular cylinder, tangent to H, axis K(1+¾−2) L(4+¾−½).

581. Find the intersection, and its true size, of the plane W(4+3) 1(4−3) and the cylinder in Exercise 580.

582. Find the intersection, and its true size, of the plane P(1+3) 4(1−2) and the 1½-in. circular cylinder, whose axis is A(1+1−1) B(5+1−1).
583. Find the true size of the right section of an elliptical cylinder, whose axis is C(3, 0−2) D(3+3, 0), and whose base is a 1½-in. circle in *H*. Find its intersection with *V*.
584. Find the intersection, and its true size, of the plane R(+3)∞(−2) and the 1½-in. circular cylinder, whose axis is E(3+1−1) F(3+2−3). Find also the *H* and *V* intersections of the cylinder.
585. Find the *H* and *V* intersections of the 1½-in. circular cylinder, whose axis is G(3, 0−2) H(3+1½, 0).
586. Find the *H* and *V* intersections of the 1-in. circular cylinder, whose axis is K(2+½−2) L(4+1½−½).
587. Find the points in which A(1+2−3) B(3+1−1) pierces the cylinder in Exercise 575.
588. Find the points in which C(2+3−½) D(3+1−2) pierces the cylinder in Exercise 578.
589. Find the points in which E(2+2−2) F(4+½−½) pierces the cylinder in Exercise 585.
590. Find the shortest line that can be drawn from G(1½+1, x) on the rear surface to *H*(2¼+2, y) on the front surface of the cylinder in Exercise 573. What is this line?
591. Find the projections of the shortest line that can be drawn on the surface of the cylinder in Exercise 580, from M(1½, x−1¼) on the under to N(3, y−1¼) on the upper surface. What is this line?
592. Draw the shortest line from O(3½, +1, x) on the rear to P(2¼+1½, y) on the front surface of the cylinder in Exercise 583.
593. Develop the surface of the cylinder in Exercise 573, between *H* and T.
594. Develop the surface of the cylinder in Ex. 574 between *H* and S.
595. Develop the surface of the cylinder in Ex. 583 between *H* and *V*.
596. Develop the surface of the cylinder in Ex. 575 between *H* and W.
597. Develop the surface of the cylinder in Ex. 584 between the base at E and the plane R.
598. Develop the surface of the cylinder in Ex. 584 between *H* and R.
599. Cylinder in Ex. 575. Draw the traces of a tangent plane through A(2+1, x) on the front surface.
600. Cylinder in Ex. 575. Draw the traces of a tangent plane through the point B(2½+1−1).
601. Cylinder in Ex. 575. Draw the traces of a tangent plane that is parallel to M(5+2−2) N(4, 0−1).
602. Cylinder in Ex. 575. Draw the traces of a tangent plane inclined 75° to *H*.
603. Cylinder in Ex. 578. Draw the traces of a tangent plane containing E(2+1, x) on the front surface.
604. Cylinder in Ex. 578. Draw the traces of a tangent plane containing F(4+1−½).
605. Cylinder in Ex. 578. Draw the traces of a tangent plane parallel to K(5+2−1) L(5½+1−¼).
606. Cylinder in Ex. 578. Draw the traces of a tangent plane inclined 45° to *H*.

607. Cylinder in Ex. 580. Draw the traces of a tangent plane containing A($2\frac{1}{2}+\frac{3}{8}$, x) on the rear surface.
608. Cylinder in Ex. 580. Draw the traces of both tangent planes through B($3+2\frac{1}{2}-1\frac{1}{4}$), and measure the angle between them.
609. Cylinder in Ex. 580. Draw the traces of the four tangent planes inclined 60° to H.
610. Cylinder in Ex. 580. Draw the traces of both tangent planes parallel to C($1+2\frac{1}{2}-2\frac{1}{2}$) D($1\frac{1}{4}+3-3$).
611. Cylinder in Ex. 582. Draw the traces of the tangent plane containing E($2+\frac{3}{8}$, x) on the lower rear surface.
612. Cylinder in Ex. 582. Draw the traces of both tangent planes through the point F($3+2\frac{1}{2}-\frac{1}{2}$).
613. Cylinder in Ex. 582. Draw the traces of both tangent planes parallel to G($4+2-3$) H($4\frac{1}{2}+2\frac{3}{4}-2$).
614. Cylinder in Ex. 582. Draw the traces of the four tangent planes inclined 45° to H.
615. Cylinder in Ex. 583. Draw the traces of the tangent plane containing K($3\frac{1}{2}+1$, x) on the lower surface.
616. Cylinder in Ex. 583. Draw the traces of the tangent plane containing M($4\frac{1}{2}+1-1$).
617. Cylinder in Ex. 583. Draw the traces of a tangent plane inclined 60° to H.
618. Cylinder in Ex. 583. Draw the traces of a tangent plane parallel to N($1+1-1$) O($2, 0-2\frac{1}{2}$).
619. Cylinder in Ex. 584. Draw the traces of the tangent plane through P($3\frac{1}{2}+2$, y) on the upper surface.
620. Cylinder in Ex. 584. Draw the traces of a tangent plane through A($4\frac{1}{2}+1-1$).
621. Cylinder in Ex. 584. Draw the traces of two tangent planes inclined 75° to H.
622. Cylinder in Ex. 584. Draw the traces of a tangent plane parallel to B($1+1-1$) C($2, 0-2\frac{1}{2}$).
623. Cylinder in Ex. 586. Draw the traces of the tangent plane containing D($3+\frac{3}{4}$, x) on the rear surface.
624. Cylinder in Ex. 586. Draw the traces of a tangent plane containing E($3\frac{1}{2}+\frac{1}{2}-2$).
625. Cylinder in Ex. 586. Draw the traces of a tangent plane inclined 45° to H. Use the "Normal method."
626. Cylinder in Ex. 586. Draw the traces of a tangent plane parallel to F($1+1-1$) G($2, 0-2\frac{1}{2}$). "Normal method."
627. Draw the projections of a $1\frac{1}{2}$-in. circular cylinder tangent to the plane T($1+2\frac{1}{2}$) 4($2\frac{1}{2}-1\frac{1}{4}$) along the line A($1+2\frac{1}{2}$, 0) B($2\frac{1}{2}$, $0-1\frac{1}{4}$).

PRACTICAL EXERCISES

Note.—Some of these exercises combine the cylinder and cone. When pieces are duplicated, develop only one.

628. Develop the patterns for the two-piece 8-in. pipe elbow (Fig. 92). Any convenient scale.

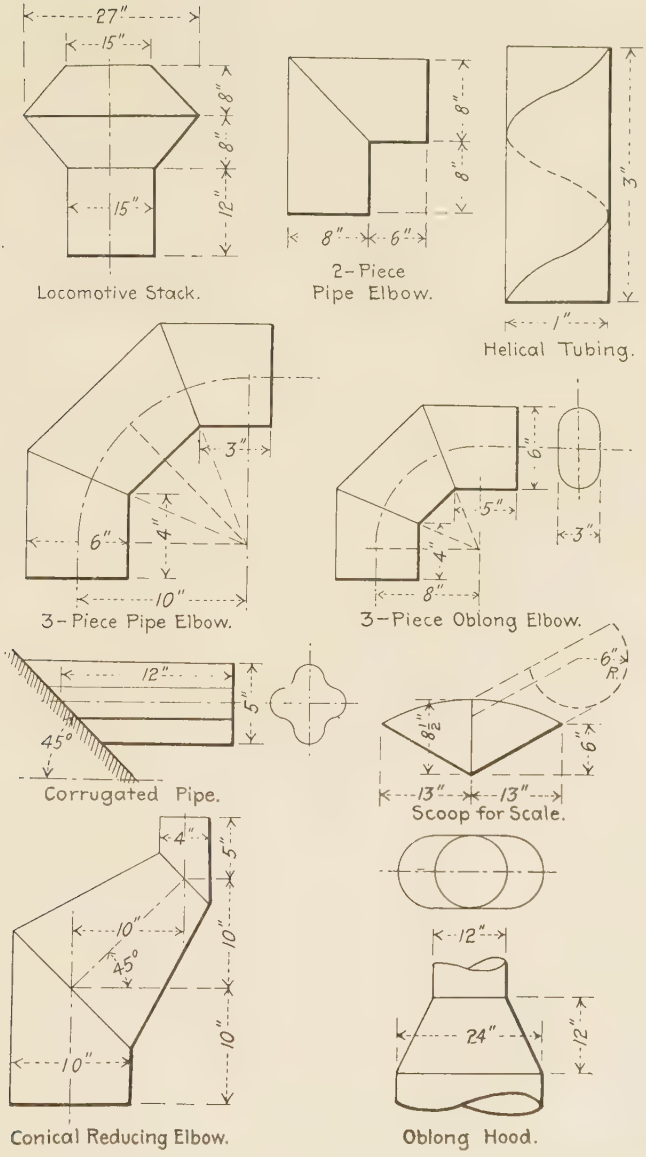


FIG. 92.—Practical examples of cylindrical and conical surfaces.

629. Develop the patterns for the Locomotive Stack in Fig. 92. Any convenient scale.
630. Lay out the patterns for the 3-piece 6-in. pipe elbow (Fig. 92). This pipe is composed of circular cylinders. Figure the angles for the joints so as to keep the diameter uniform.
631. Design a 4-piece 6-in. pipe elbow for a 90° turn, and lay out the patterns to scale.
632. Design a 4-piece 8-in. pipe elbow for a 105° turn, and lay out the patterns to scale.
633. Design a 5-piece 6-in. pipe elbow for a 90° turn, and lay out the patterns to scale.
634. Lay out the patterns for the oblong elbow (Fig. 92) to scale.
635. Design a 4-piece oblong elbow, of the same section as in Fig. 92 (Ex. 634), for a 90° turn, and lay out the patterns.
636. Design a 4-piece oblong elbow, of the same section as in Fig. 92 (Ex. 634), for a 120° turn, and lay out the patterns.
637. Lay out the patterns for the piece of helical tubing (Fig. 92). Full size.
638. Lay out the patterns to scale for the scale scoop (Fig. 92).
639. Lay out the patterns for the conical reducing offset elbow (Fig. 92). The 4-in. and 10-in. pipes connected by the elbow are both of circular section, and the bases of the cone are perpendicular to its axis. Develop the three pieces to scale.
640. Lay out the patterns to scale for the corrugated galvanized iron pipe (Fig. 92). Design your own sectional outline.
641. Lay out the pattern to scale for the oblong reducing hood (Fig. 92).

Note.—Use two spaces for all these exercises, except 637.

THE CONVOLUTE

189. The Convolute, the third class of single-curved surfaces, is generated by the tangent line to any space curve, as it moves along the curve. The tangent line in any position contains two consecutive points of the curve (see Definition, Art. 118). This means that any two consecutive tangents to any curve have one point in common; thus, A, B and C are three consecutive points in a curve, and AB and BC are therefore two consecutive tangents with B as their intersection. Thus all **consecutive tangents** to any space curve intersect, and therefore become elements of a single-curved surface. The convolute is, therefore, a single-curved surface.

190. Helical Convolute.—The most important surface of this class is that generated by the tangent to the helix. Fig. 93 shows both projections. It is sometimes called the "Developable Helicoid" and is found in practical use in some designs of screw conveyors.

The surface is limitless, the portion here shown being only a

"three-quarter turn." In practical design it is always limited by a cylinder and upper and lower bases. The H-trace of the surface is an involute, and any plane perpendicular to the axis will cut the surface in an involute of a circle whose diameter is that of the helical cylinder.

Note.—In all the problems relating to convolutes the axis will be taken perpendicular to H, as in Fig. 93.

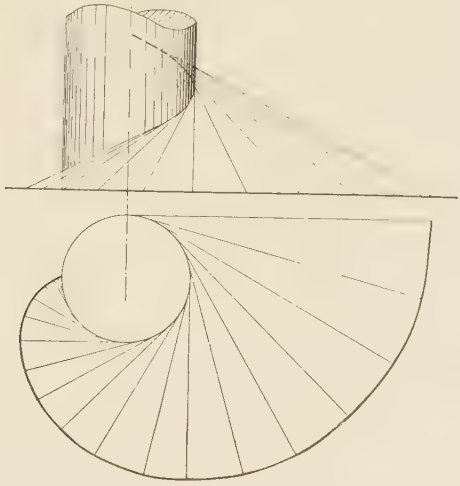


FIG. 93.—Projections of a convolute.

191. Problem 77. —**To locate a point on the surface of a convolute, one projection being given.**

First Case.—When the H-projection is given.

Analysis.—1. Through the given projection draw the tangent to the circle, which is the H-projection of the helix, intersecting the involute trace.

2. Draw the V-projection of this tangent, by projecting the point of tangency to the V-projection of the helix, and the involute intersection to GL.

3. Having both projections of the element through the point, locate on it the V-projection of the point.

Second Case.—When the V-projection is given.

Analysis.—1. Pass a plane through the point, parallel to H, and obtain its intersection (involute) with the surface.

2. Project from the V-projection of the point to the involute, thus obtained, to locate the H-projection.

192. Problem 78.—To pass a plane tangent to a convolute through a point on the surface.

1. Draw the element through the point. This is one line of the required tangent plane.

2. Where the element of contact intersects the involute, draw the tangent. This will be the H-trace of the tangent plane.

3. Find the V-trace by any of the known methods.

193. Problem 79.—To pass a plane tangent to a convolute through a point outside the surface.

Analysis.—1. With the given point as apex, draw a right circular cone, its base on H, its elements having the same inclination to H as those of the convolute.

2. Pass a plane tangent to both cone and convolute. This will be the required tangent plane.

Second Method.—Analysis.

1. Pass a plane parallel to H through the given point.

2. Obtain its intersection with the convolute.

3. Draw the tangent from the given point to this intersection.

4. This tangent and the element of contact will determine the tangent plane.

194. Problem 80.—To find the intersection of a convolute by a plane.

Analysis.—Find the piercing points of a sufficient number of elements with the given plane and draw a smooth curve through these points.

Note.—As the surface ends at the directrix curve (practically, if not theoretically), any piercing points that lie beyond the points of tangency are not considered. A plane that contains an element will cut from the surface a straight line until it passes the point of tangency of that element, when it will begin cutting a curve.

195. Problem 81.—To develop the surface of a convolute.

Discussion.—If the helical directrix be rolled into a plane, element by element, it will be seen to have a **constant radius of curvature,** and will therefore develop as a circle. This radius is the normal line at any point of the helix, that intersects the axis, because all such normals will have equal lengths. The elements of the convolute, being tangent to the helix, are perpendicular to the respective normals, and when the rolling out occurs, these elements will be consequently tangent to the circle made by developing the helix. As the lengths of the tangents

are proportional to their H-projections, the developed curve of their extremities will be the involute to the developed helix. This is so, because each tangent is equal in length to the helix from the origin to the point of tangency, and when helix is developed into a circle, the tangents are equal to the arc from the origin to the point of tangency. This corresponds to the generation of the involute.

Analysis.—The entire analysis is practically confined to the determination of the radius of the circle of the developed helix.

Graphical Solution.—Take the tangent BF which is parallel to V, Fig. 94. At the point d′, where b′f′ crosses the V-projection

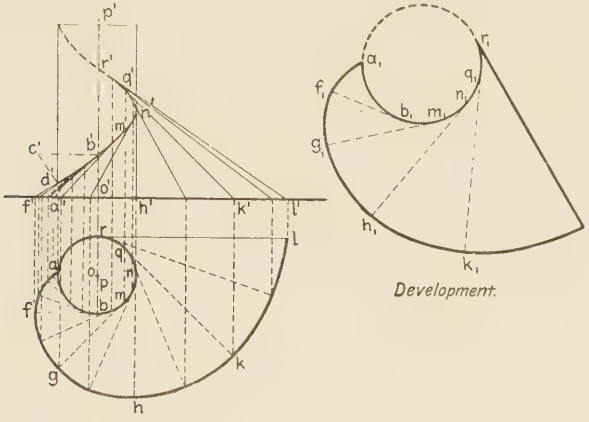


Fig. 94.—The convolute and its development.

of the outside element of the cylinder, draw c′d′ perpendicular to f′b′, and at b′ draw a horizontal line b′c′ intersecting the perpendicular c′d′ at c′. The required radius is then c′b′.

Construction.—1. Draw the circle of radius equal to c′b′.

2. Lay off arcs a_1b_1, a_1m_1, a_1n_1, etc., making them equal to the **true lengths** of the corresponding tangents to the helix.

3. Draw the involute from a_1 to r_1, and the plane figure included within the involute is the development.

Mathematical Value.—The mathematical value for the radius of the development is the expression $\frac{R}{\cos^2.\theta}$, in which R is the radius of the helical cylinder, and θ is the inclination of the elements to the horizontal.

196. EXERCISES

642. Draw a 270° convolute to the helix, whose axis is A($1\frac{1}{2}$, $0-1\frac{1}{4}$) B($1\frac{1}{2}+2-1\frac{1}{4}$), 1-in. diameter, 2-in. pitch. Find the traces of the tangent plane containing C($2\frac{1}{2}$, $x-2$) on the surface.

643. Draw the convolute in Ex. 642. Draw the traces of the tangent plane through D($1\frac{3}{4}+\frac{1}{4}$, y) on the front surface.

644. Draw a 270° convolute to the "left-hand" helix, whose axis is E($4\frac{1}{2}$, $0-1$) F($4\frac{1}{2}+3-1$), $1\frac{1}{8}$-in. diameter, $2\frac{1}{2}$-in. pitch. Draw the traces of a plane tangent at the point G($4\frac{3}{4}$, $x-1\frac{3}{4}$) on the surface.

645. Draw the convolute in Ex. 644. Draw the traces of the plane tangent at K($3\frac{1}{2}+1$, y) on the front surface.

646. Draw the convolute in Ex. 642. Draw the traces of a plane tangent containing M($1+2-2\frac{1}{2}$).

647. Convolute in Ex. 644. Draw the traces of the tangent plane containing N($4\frac{1}{4}+1-2\frac{1}{2}$).

648. Convolute in Ex. 642. Find its intersection with the plane S($1+1$) $4(1-3)$.

649. Convolute in Ex. 644. Find its intersection with the plane T($+1\frac{1}{2}$) $\infty(-3)$.

650. Develop the surface of the convolute in Ex. 642.

651. Develop the surface of the convolute in Ex. 644.

652. Find the shortest path on the surface of the convolute in Ex. 642 from O($1\frac{3}{4}$, $x-2$) to P(3, $y-1$).

653. Find the shortest path on the surface of the convolute in Ex. 644 from C(5, $x-1\frac{3}{4}$) to D(3, $y-\frac{1}{2}$).

197. General Notes and Rules on Single-Curved Surfaces.

1. A tangent plane to any single-curved surface is tangent at every point on the element.

2. A tangent plane to a cylinder is always parallel to the axis, and to all elements.

3. A tangent plane to a cone must contain the apex, and intersects every element but the element of contact.

4. **To pass a plane tangent to a cylinder,** draw a tangent line and a line intersecting the tangent, parallel to the axis. These two lines determine the required plane.

5. **To pass a plane tangent to a cone:** a tangent line and the apex are sufficient to determine it.

6. The element of contact and a tangent line **at any point of it** determine the tangent plane to any single-curved surface.

7. Tangent planes to certain single-curved surfaces will cut the surface, but not at the point of tangency.

8. To cut an element from a cone, pass a plane through the apex.

9. To cut an element from a cylinder pass a plane parallel to the axis.

10. To cut an element from a helical convolute, pass a plane tangent to the cylinder of the helix.

11. In the development of a cone, all the elements radiate from the apex. Only a right circular cone will roll out in a segment of a circle. All other cones must be developed by triangulation.

12. In the development of a cylinder, all the elements roll out parallel. Only a right section will roll out in a straight line.

WARPED SURFACES

198. Discussion.—According to the definition in Art. 152, a warped surface is generated by moving a straight line so that no two consecutive positions shall be in the same plane. It thus falls into the classification with single-curved surfaces as a **ruled surface.** Planes may be passed through any point of a warped surface that will cut straight lines from the surface, always one, and, in the case of two surfaces, two straight lines. Other planes cut curves from them.

Warped surfaces are of many varieties and are common in practical work. In many cases, when they occur in practical work, it is not necessary for the draftsman to recognize their peculiarities, but very often, as in arches, sheet metal work, propeller blades, skew gears, conveyors, etc., it is highly important that engineers and draftsmen know their properties and methods of generation.

199. Methods of Generation.—Warped surfaces may be generated:

1. By revolving one line about another not in the same plane.

2. By moving a straight line, the **generatrix,** along two lines, **directrices,** either straight, curved,or both, keeping parallel to some plane, called the **Plane Director.**

3. By moving a straight line along three directrices, straight, curved, or both.

4. By moving a straight line along an axis at a constant angle with the axis, following a space curve as a directrix.

Other methods might be invented, but these cover all the cases met in practical work.

200. Tangent Planes.—In Art. 155 we find the general rule for tangent planes to all **Ruled Surfaces,** as follows: **One tangent**

line and the element at any point determine the tangent plane at that point.

In general, a tangent plane is only tangent **at one point** of a warped surface. There are certain points, as in conoids and warped cones, where the tangent plane will be tangent all along the element, but such exceptions will be readily recognized.

All planes tangent to warped surfaces (except in the cases just noted) intersect the surface, because they are not tangent all along the element, and, conversely, **any plane that intersects a warped surface in an element will,** in general, **be tangent to the surface at some point.**

201. Double-Ruled Surfaces.—Two surfaces, the **Hyperboloid of Revolution of One Nappe,** and the **Hyperbolic Paraboloid,** are called double ruled surfaces from the fact, as will be shown later, that they are capable of generation by **two different straight lines.** This yields the phenomenon, that **through every point in the surface two straight lines** (elements) **can be drawn lying wholly within the surface.** From this fact we have the special rule for tangent planes to these surfaces; **The two elements of a double-ruled surface through any point of the surface determine the tangent plane at that point.**

202. Representation.—Some warped surfaces can be shown **in outline** in orthographic projection with sufficient clarity, and others must be shown by drawing a sufficient number of elements. In practical drafting, where the surface is always limited, the outlines are usually sufficient.

203. Location of Points.—To locate a point on any warped surface, one projection being given, find the piercing point of the **projector** of the point with the surface. This is done by passing a plane through the projector, and finding its intersection with the surface. Where the projector intersects this line will be the location of the point. In many cases it will be easy to draw the element through the point, simplifying the operation.

204. Developments.—Strictly speaking, a warped surface **cannot be developed,** because no two consecutive elements lie in the same plane, and cannot, therefore, be rolled into a plane without "warping" them; that is, destroying the relation between them. However, there are **approximate** developments that give very satisfactory results, almost as good as the development of a cone by triangulation. These warped develop-

ments are obtained also by triangulation. A sufficient number of elements are drawn, dividing the surface into minute warped quadrilaterals, in which the amount of warp is so slight as to be negligible. Diagonals are then drawn across each of these quadrilaterals, dividing them into triangles. These triangles are then measured and placed in order in their (approximate) true size. The resulting surface, obtained by bending back this development, cannot be the exact surface, as shown in the drawing, but will be near enough for practical purposes. In making single-curved surfaces from flat patterns there need be no distortion of the metal, but in making up a warped surface, there will be some distortion.

THE HELICOID

205. The most frequent warped surface encountered in drafting, is the **Helicoid.** It is generated by Method 4, Article 199, by moving a straight line along a helix and its axis, keeping a constant angle with the axis. The helix and its axis are the directrices. The surface is theoretically unlimited, but in practise is usually included between two helices. Any circular cylinder, having its axis coincident with that of the helical directrix, will intersect the helicoid in a **helix.** Any plane perpendicular to the axis will cut the helicoid in a **spiral of Archimedes.** Any plane passed through the axis will intersect the helicoid in an **element.**

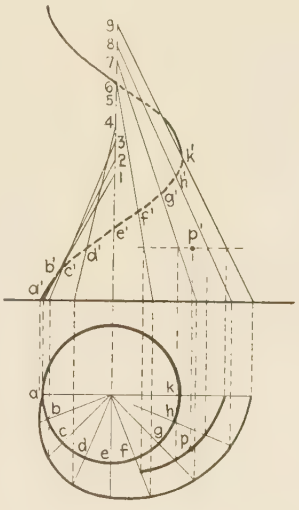


FIG. 95.—Helicoid.

Fig. 95 shows a helicoid, with its axial angle 30°. In drawing the helix a certain number of points were laid out, and through these points lines intersecting the axis are drawn, making the given angle with the axis. As the helix is everywhere equally distant from the axis, this is simple; as follows.

1. Draw one element parallel to V, which will be drawn in its true angle with the axis.

2. As the generatrix is moved, it climbs both axis and helix in exactly the same upward distances. Therefore, from any

point on the helix, an element may be drawn to a point on the axis that is the same distance up, measured on the V-projection. Thus in Fig. 95, the point b′ is the same distance above a′ as the point 2 is above 1 on the axis.

The point P is located on the surface by drawing the element through the point and locating the other projection on the element. If the V-projection of the point is given, pass a plane through the point parallel to H. This will cut the elements in a curve on which the H-projection will be located.

The helicoid is the surface of the V-thread, the square thread, spiral springs, screw conveyors, certain fire-escapes, the twist drill, the spiral runway, helicoidal arch, and numerous other surfaces frequent in machine and structural design.

Helicoidal arch.—In the design of skew arches, that is, arches with their axes oblique to one or both of the openings, warped surfaces are often encountered, **cylindroids** or **cow's horns** being the usual surfaces in such arches. The arch called the **helicoidal arch** takes its name not from the character of its surface, as might be supposed, but from the surface of the "coursing joints." The line of the joint is laid out as a straight line on the developed surface of the arch, and, when brought into its actual position, it becomes a helix. The surface of the joint is generated by a line normal to the surface of the arch moving along the helix, radial to the axis of the helix. This surface is therefore identified as a helicoid. Inasmuch as the developed helix is a straight line the work of laying out the blocks is very simple. The seam, being helicoidal, must be laid out with care, to ensure good fitting blocks.

THE HYPERBOLOID OF REVOLUTION OF ONE NAPPE

206. This peculiar surface is the only warped surface that is a **surface of revolution.** It is generated by Method 1, Article 199, by revolving a straight line about an axis not in the same plane. Fig. 96 shows this surface represented by its elements. Either projection of all the elements could be inverted, and there would be no change in the surface. If AB be considered an element, its V-projection might be either of the lines marked a′b′, and the same surface would be generated by revolving either line about the axis. This makes it a **double-ruled surface,** and a tangent plane is determined at any point by two elements through that point.

207. To draw the Elements.—All points in the generatrix move in circles, hence there is a vertex to the curve of the outline, a smallest circle, called the **Gorge Circle,** generated by the point nearest the axis. All elements intersect the gorge circle and their H-projections are all tangent to its H-projection. Two limiting planes, **bases,** are always necessary, always taken perpendicular to the axis, and usually at equal distances from the gorge circle. Being perpendicular to the axis, these bases are circular. Why? One base is usually in H, and the elements

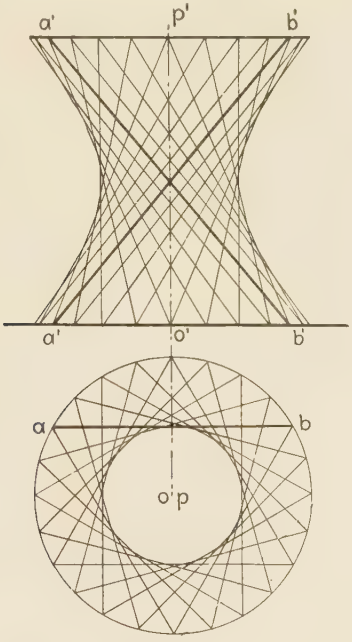


Fig. 96.—Hyperboloid of revolution.

terminate in the bases. Hence, 1. Draw the gorge and base circles. 2. Draw as many elements as desirable, 16 or 24, the H-projections tangent to the gorge circle and terminating in the base circles. 3. Project the terminals of the elements to the V-traces of the respective base planes.

Note.—When the bases are symmetrically placed, there will be only one circle projected on H, as in Fig. 96.

208. To Locate a Point on the Surface.—If the given projection is on H, draw an element through the given projection and project the point to the V-projection of the element.

If the given projection is on V, pass a plane through the surface, parallel to H, through the given projection. This will cut a circle from the surface, projected in its true size on H. The H-projection will be in the circle.

209. To Pass a Plane Tangent at any Point on the Surface.—Draw the two elements through the given point. They will determine the tangent plane.

Tangent planes to this surface, as to all surfaces of revolution, are perpendicular to the plane of the axis and the point of tangency, called the **Meridian Plane.**

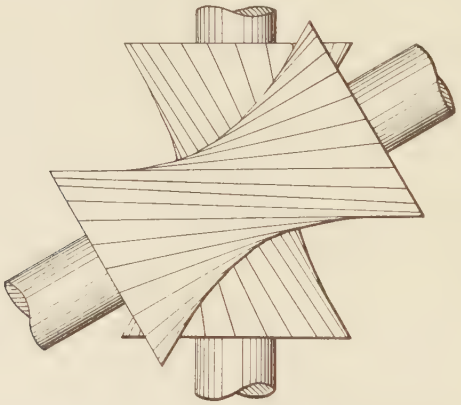


FIG. 97.—Rolling hyperboloids. (Both surfaces are generated by revolving the same straight line about two different axes, and they thus have one element in common, and are tangent to each other along this element.)

The **Meridian Section** of this surface is a hyperbola, and the same surface can be generated by revolving a hyperbola about its vertical axis. If proof of this is desired, it can be found in Church and Bartlett, page 150.

The most important function of the hyperboloid is found in the pitch surfaces of skew gears. If one line be successively revolved about two axes, not in its plane, the line will be simultaneously an element of both hyperboloids; that is, the two hyperboloids generated will be tangent. These, if used as the basis of gears, form an ideal transmission medium through shafts not in the same plane.

THE HYPERBOLIC PARABOLOID

210. This surface is generated by moving a straight line along two straight lines, not in the same plane, keeping it always

parallel to a plane director. The elements are all parallel to this plane, but not to each other.

Representation.—If a definite position of the two directrices is given, the elements may be drawn without the assistance of the plane director, by dividing each directrix into the same number of equal parts and connecting the points in order. This is so, because a series of parallel planes will cut any two lines in proportional segments.

211. Second Generation.—If any two elements be taken, as BD and AC (Fig. 98) and divided as were the directrices AB and CD, and joined in order, the surface thus generated would be

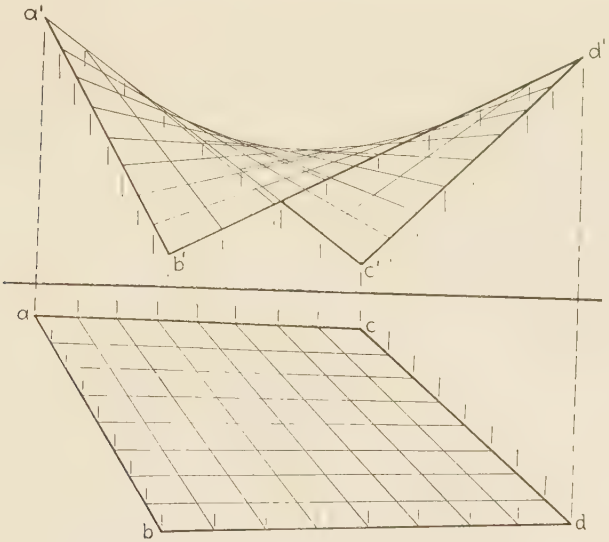


Fig. 98.—Hyperbolic paraboloid, showing double ruling.

identical with the first surface. In the two generations, the directrices of the first become elements of the second, and the elements of the first become directrices. This makes it a double-ruled surface, and the rule for tangents is the same as for the hyperboloid.

212. It is named the Hyperbolic Paraboloid because certain planes cut hyperbolas from the surface, others parabolas, while tangent planes cut straight lines. It is chiefly of interest to the mathematician, but it is occasionally met in practical work, and is then usually called a "Warped Quadrilateral."

THE CYLINDROID

213. The Cylindroid is a warped surface with two curved-line directrices and a plane director. It differs from a cylinder in that its elements, while parallel to one plane are not parallel to each other. It is found in practical work in arches and sheet metal work.

THE CONOID

214. The Conoid has one curved-line directrix, one straight-line directrix, and a plane director.

A helicoid, whose elements are at right angles to the axis, may be regarded as a conoid.

THE COW'S HORN

215. This surface has three directrices, two being circles, and the third a straight line. The circles are usually in parallel planes with their centers in a plane perpendicular to these parallel planes. The straight line directrix usually lies in the plane of the centers. The Cow's Horn is frequently encountered in stone arch designs, and occasionally in sheet metal hoods.

THE WARPED CONE

216. This surface, often found in reducing hoods and pipe connections, has three directrices; as follows, two circles or ellipses (or one of each) in planes not parallel, and the line joining their centers, called the axis.

217. Other Warped Surfaces.— Besides the foregoing, there are many forms, which have no specific names. They may have three straight directrices, three curved directrices, or combinations of these with plane directors. Whenever they are met, they will present no difficulties to the student, who is familiar with those described.

218. The Development of a Specimen Warped Surface. Develop the helicoid shown in Fig. 99, for a half turn, between *H* and the helical directrix.

1. Draw the elements A1, B2, etc.

2. Draw diagonals A2, B3, etc., of the quadrilaterals.

3. Measure the true lengths of the various elements and diagonals. The construction on the left shows a simple arrangement for accomplishing this. Lay off the heights of the various points, a′, b′, etc., on a vertical line, and lay off the lengths of

the various H-projections on a horizontal line, elements on one side, diagonals on the other. Their true lengths will be the hypothenuses, a′1 or a′2, etc.

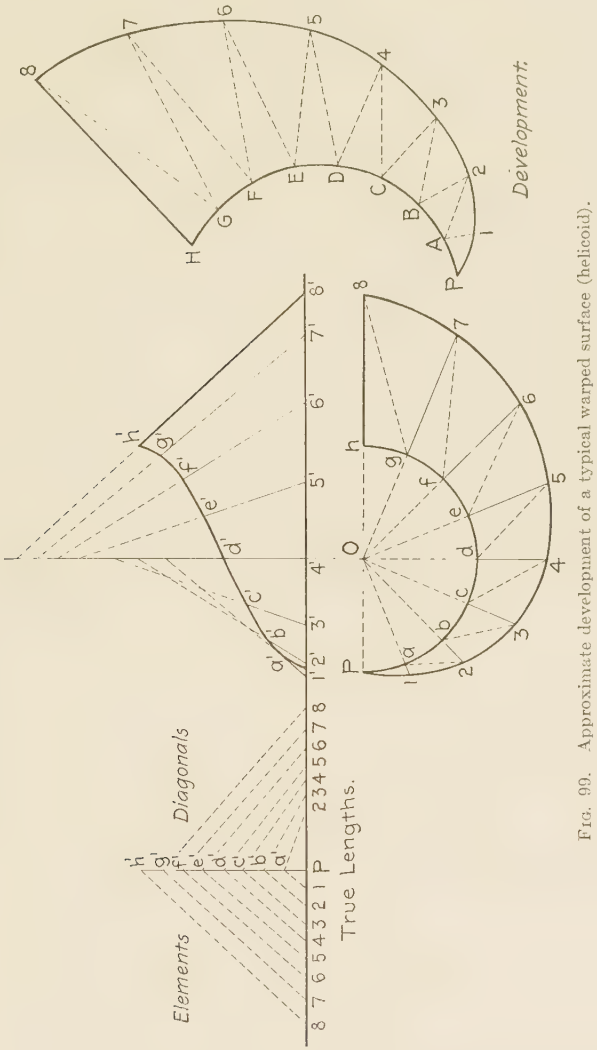


Fig. 99. Approximate development of a typical warped surface (helicoid).

4. Rectify the various arcs of the helix and spiral. The arcs of the helix are equal in length to corresponding arcs of a circle whose radius $= \frac{\text{OP}}{\cos^2\theta}$. (See the graphical solution given in Art.

196.) The lengths of the spiral arcs are obtained by stepping off small chords with the dividers.

5. With these materials, construct the triangles in order, as shown on the right.

This method, with such modifications as may be necessary, is employed in the development of all warped surfaces. It is laborious, but there is no shorter method. The important operation in this, as in all problems relating to surfaces, is to get the elements.

219. EXERCISES IN WARPED SURFACES

654. Draw the projections of a right-hand helicoid, its axis A(2, 0−1½) B(2+3−1½), and its helical directrix of 1½-in. diameter and 2½-in. pitch. The constant angle between elements and axis is 45°. Draw the *H*-trace of the helicoid, and locate C(2¼+1, x) and D(3, y−2½) on the surface.

655. Draw the projections of a left-hand helicoid, its axial directrix being M(4, 0−1¼) N(4+3−1¼), its helical directrix having a 1¼-in. diameter and 2-in. pitch. The constant angle between elements and axis is 60°. Draw the *H*-trace of the helicoid, and locate O(4¾+⅜, x) and P(3, y−2) on the surface.

656. Helicoid in Ex. 654. Draw the traces of the tangent plane at C(2¼+1, x) or D(3, y−2½) on the surface.

657. Helicoid in Ex. 655. Draw the traces of the tangent plane at O(4⅜+⅜, x) or P(3, y−2) on the surface.

658. Helicoid in Ex. 654. Find the points in which it is pierced by the line E(1+¾−1)F(5+¾−3).

659. Helicoid in Ex. 655. Find the points in which it is pierced by the line K(2+1½, 0)L(5½, 0−3).

660. Helicoid in Ex. 654. Find its intersection with the plane Q(3+3) 3(3−3).

661. Helicoid in Ex. 654. Find its intersection with the plane R(1+1) 5(1−3).

662. Helicoid in Ex. 655. Find its intersection with the plane S, parallel to *H*, ⅜ in. above *H*.

663. Helicoid in Ex. 655. Find its intersection with the plane T(2+2) 2(5 3½).

664. Draw the projections (24 elements) of a hyperboloid of revolution, its axis being A(3, 0−1½) B(3+3−1½) and one element being C(2+2½−¾) D(4, 0−¾). Locate E(2½+½, x) on the rear surface, and draw the traces of the plane tangent at E.

665. Draw the projections (24 elements) of a hyperboloid of revolution, axis K(3+1½, 0) L(3+1½−3), one element M(2+¾−2½) N(4+¾, 0). Locate O(3½, y−1) on the under surface, and draw the traces of the tangent plane at O.

666. Hyperboloid in Ex. 664. Find its intersection with the plane Q(3½+3) 3½(3½−3). Show profile view.

667. Hyperboloid in Ex. 665. Find its intersection with the plane R(2+3) 2(5−2). Show its true size.

668. Hyperboloid in Ex. 664. Find its intersection with the plane S(+1) ∞(−3). Show its true size.

669. Hyperboloid in Ex. 665. Find its intersection with the plane T(2+3) 5(2−2). Show its true size.

670. Hyperboloid in Ex. 664. Find the points where it is pierced by the line F(1+1−1) G(5+1−3).

671. Hyperboloid in Ex. 665. Find the points in which it is pierced by A(1+3, 0) B(5, 0−2).

672. Hyperboloid in Ex. 664. Do not draw the elements. Pass a plane through the line F(2+1, 0) G(5+1−2) that will be tangent to the hyperboloid and find the point of tangency. (Hint: A plane through the axis, perpendicular to the plane through FG, will contain the point of tangency, Art. 209. Also see Art. 200.)

673. Hyperboloid in Ex. 665. Pass a plane through the line A(1+3, 0) B(4, 0−2) that will be tangent to the hyperboloid and find its point of tangency. (Hint: See Art. 208 and 209.)

674. Let A(2, 0−2) B(2+3−2) and C($\frac{1}{2}$+1−1) D(3$\frac{1}{2}$+1−1) be the axes of two hyperboloids having a common element E(2$\frac{3}{4}$+2−1$\frac{3}{8}$) F(1$\frac{1}{4}$, 0−1$\frac{3}{8}$). Draw the projections, 24 elements, of each hyperboloid, and specify the faster revolving in transmitting power.

675. A($\frac{1}{2}$+3−1$\frac{1}{2}$) B(1$\frac{3}{4}$+1−$\frac{1}{2}$) and C(2$\frac{3}{4}$+1−2$\frac{1}{2}$) D(4$\frac{1}{2}$+3−1) are directrices of a hyperbolic paraboloid, and T(4+1) 4(5−$\frac{1}{4}$) is its plane director. Draw ten elements of the surface and locate a point E (2+x−y) on the surface between elements 3 and 4.

676. K($\frac{1}{2}$+1−2) L(2+3−$\frac{1}{2}$) and M(3$\frac{1}{2}$+2$\frac{1}{2}$−2$\frac{1}{2}$) N(4$\frac{3}{4}$+1−1) are directrices of a hyperbolic paraboloid, and KM and LN are elements. Draw eight elements, and find the traces of a plane tangent at O(2 +2$\frac{1}{4}$, x) on the surface.

677. A($\frac{1}{2}$+3−1$\frac{1}{2}$) B(1$\frac{3}{4}$+1−$\frac{1}{2}$) and C(2$\frac{3}{4}$+1−2$\frac{1}{2}$) D(4$\frac{1}{2}$+3−1) are directrices of a hyperbolic paraboloid, of which AC and BD are elements. Find the traces of a tangent plane at the point E(2+x−y) on the element midway between AC and BD. Do not try to show the surface by elements.

678. A(2, 0, 0) and B(3$\frac{1}{2}$, 0−2) are the centers of two 3-in. semicircles, lying respectively in *V* and T(5+2) 5(3$\frac{1}{2}$−2). They are directrices of a cylindroid having *H* as its plane director. Show the surface by 12 elements, and find the traces of a plane tangent at C(3, x−$\frac{1}{2}$) on the surface.

679. A cow's horn has for directrices a 3-in. semicircle in *V*, center at D (2, 0, 0), and a 1$\frac{1}{4}$-in. semicircle parallel to *V*, center at E(3, 0−$\frac{3}{4}$), and the line M(2$\frac{1}{2}$, 0, 0) N(2$\frac{1}{2}$, 0−3). Show the surface by 12 elements, and draw the traces of a plane tangent at F(3+x−$\frac{3}{8}$) on the surface.

680. Cow's horn in Ex. 679. Locate a point on the surface 1 in. above *H* and $\frac{1}{2}$ in. in front of *V*. Draw the traces of a plane tangent at the point.

681. The directrices of a conoid are a 2-in. circle in *H*, center at M(2, 0−2) and the line N(2$\frac{1}{2}$+2−2) P(5+2−2). The plane director is perpen-

dicular to V and inclined 60° to H. Show the surface by 24 elements, and find the traces of a plane tangent at a point on the surface $1\frac{1}{2}$ in. from V and $\frac{1}{2}$ in. above H.

682. Conoid in Ex. 681. Intersect it by the plane $T(1+1\frac{1}{4})\ 5(2-3\frac{1}{2})$. Show the true size of the intersection.

683. Develop the surface by triangulation of the conoid in Ex. 682, between H and T. (682 and 683 should be given together with one space for each.)

684. A warped cone has one base a 2-in. circle in H, center at A$(2,\ 0-1\frac{1}{2})$, and one base an ellipse in $T(1+2)\ 4(4+3)$ whose H-projection is a 1-in. circle, center at A. The axial directrix is the line joining the centers of the circle and ellipse. Show its surface by 16 elements, and find the traces of a plane tangent at $B(x+\frac{3}{4}-1)$ on the surface.

685. Develop the warped cone in Ex. 684. (When this exercise is given, 684 and 685 should be given together, one space for each.)

686. $A(\frac{1}{2}+2-2)$ $B(1\frac{3}{4}+\frac{1}{2}-\frac{1}{2})$, $C(2+1\frac{1}{2}-\frac{1}{2})$ $D(3+\frac{1}{4}-2)$, and $E(3\frac{1}{2}+\frac{1}{2}-3)$ $F(4+2\frac{1}{2}-\frac{1}{2})$ are the directrices of a warped surface. Draw the element through B.

687. In the warped surface in Ex. 686, find the projections of a point 1 in. above H and 1 in. from V.

PRACTICAL EXERCISES IN WARPED SURFACES

In all the development exercises use two spaces.

688. Draw the projections of a helical spring made of $\frac{1}{4}$-in. square steel, outside diameter 3 in., pitch $1\frac{1}{4}$ in., 2 coils.

689. Draw the projections of a left handed square thread, 2-in. diameter, $\frac{3}{4}$-in. pitch (single thread), and 3 in. long.

690. Draw the projections of a helical spring made of $\frac{1}{4}$-in. wire, outside diameter $2\frac{1}{2}$ in., pitch 1 in., and 3 coils.

691. Draw the projections of a V-thread, $2\frac{1}{2}$-in. diameter, $\frac{1}{2}$-in. pitch, 3 in. long. Show a section made by a plane perpendicular to the axis.

692. Draw the projections of a twist drill, 1-in. diameter, $5\frac{1}{2}$ in. long, made from a flat bar $\frac{1}{4}$ in. thick. Pitch of helix $3\frac{1}{2}$ in. Sharpen point at 30°. See Fig. 100. Do not use a ground line.

693. Design a screw conveyor to scale, 12-in. diameter, capable of delivering 6000 cu. ft. of grain per hour, making 100 revolutions per minute. Make the diameter of the shaft 2 in. See Fig. 100.

694. Draw a conical-helical spring, $\frac{1}{4}$-in. round wire, large diameter $2\frac{1}{2}$ in., small diameter $1\frac{1}{4}$ in., 3 turns, 1-in. pitch.

695. Draw the projections of the pitch surfaces of two hyperboloidal gears, operating at 45°, having a gorge circle ratio of 5 to 3.

Note.—Draw one axis perpendicular to H, and one inclined 45° to H. Take a line for the common element, at relative distances of 5 and 3 from the axes, bisecting their angle.

696. Show the surface of a cow's horn arch, which has one opening a 12-ft. semicircle at A$(2,\ 0,\ 0)$ and the other an 8-ft. semicircle at $B(3,\ 0-1)$. Draw the projections of a 2-ft. square stone on the interior of the arch,

its center being at C($2\frac{3}{4}+\frac{1}{4}-x$). (Hint: Bound the block with four planes, and find their intersections with the surface.) Scale $\frac{1}{4}$ in. = 1 ft.

697. Design a conoidal top-piece for the hip roof shown in Fig. 100. Make

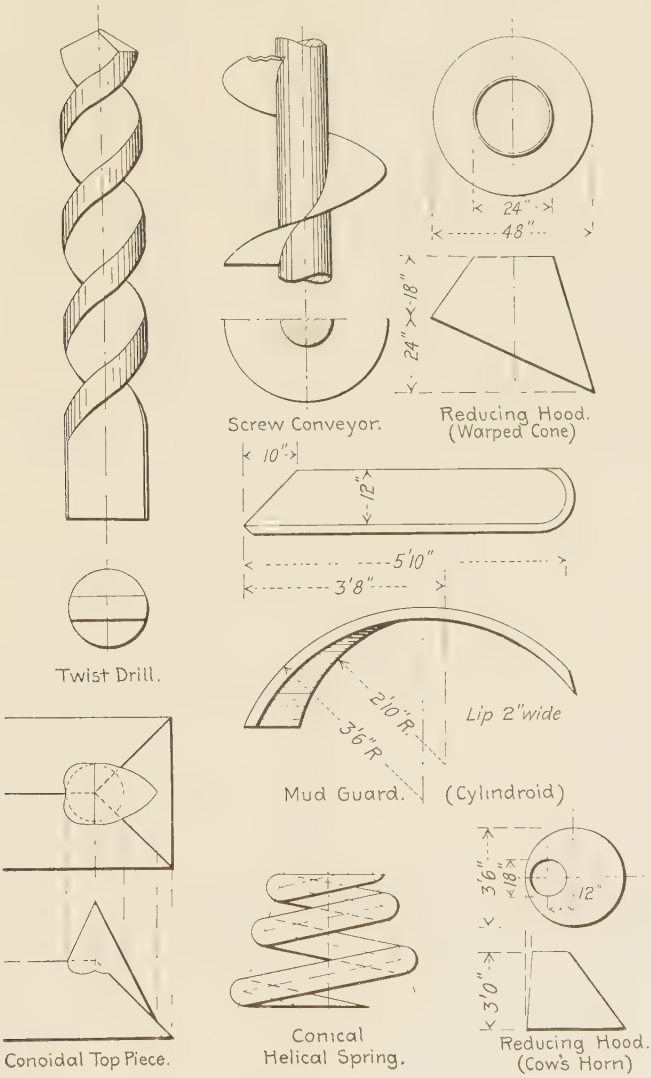


FIG. 100. –Practical warped surfaces.

all three roofs inclined 45°, and make the conoid any suitable proportions. Have the center of the circular directrix at the point of intersection of the three roof planes, the straight directrix directly over

that point, and the plane director perpendicular to the straight directrix. Obtain the roof intersections, and develop the conoid.

698. Develop the surface of the automobile mud guard (cylindroid) to scale (Fig. 100).

699. Develop the surface of the reducing hood (warped cone) shown in Fig. 100. Any convenient scale.

700. Develop the surface of the reducing hood (cow's horn) shown in Fig. 100. Any convenient scale.

DOUBLE-CURVED SURFACES

220. A double-curved surface is generated by a curve following a curve in such a way as to generate a surface that is neither single-curved or warped. A double-curved surface contains **no straight lines.** It may be either double convex, like a sphere or ellipsoid, or it may be concavo-convex, like a torus, or like the turned surfaces of numberless articles.

As all important double-curved surfaces are also **surfaces of revolution,** they will be considered under that head.

SURFACES OF REVOLUTION

221. Any line revolving about an axis generates a surface of revolution.

222. Single-Curved Surfaces of Revolution. If the generatrix is a straight line and in the plane of the axis, the surface will be single-curved. The **Cylinder of Revolution** and the **Cone of Revolution** are the only examples.

223. Warped Surfaces of Revolution. The only example in this class is the **Hyperboloid of Revolution of One Nappe,** see Art. 206.

224. Double-Curved Surfaces of Revolution.—Any plane curve revolving about an axis generates this surface.

Sphere.—Generated by a circle revolving about one of its diameters.

Ellipsoids. 1. **Prolate Spheroid.** Generated by revolving an ellipse about its major axis. Its shape resembles a foot ball.

2. **Oblate Spheroid.**—Generated by revolving an ellipse about its minor axis. Its shape resembles a door knob.

Paraboloids.—Generated by revolving a parabola about either axis. When revolved about the transverse axis, it generates the surface used in search-light reflectors. Why?

Torus.—Generated by revolving a circle about an axis that is not a diameter, usually completely outside the circle. In this case it is called a **ring,** or **annular torus.** As a matter of fact, a torus is often generated by other curves than the circle, or by a combination of curves. The architectural detail is of this character. In this work the word torus will be used to mean the annular torus.

Hyperboloid of Revolution of Two Nappes.—Generated by revolving a hyperbola about its transverse axis. It is not often encountered in drafting.

Many surfaces generated by combination curves will be found, some with names and some without.

225. Meridian and Right Sections.—Every plane perpendicular to the axis cuts a surface of revolution in a circle. Hence, we may even conceive a surface to be generated by a circle moving in the direction of the axis, and changing its diameter according to some law. Any surface turned in a lathe is generated in this way. (See Fig. 101).

Every plane through the axis, called a **Meridian Plane,** cuts the surface in a **Meridian Line,** which has the identical form of the line used to generate the surface.

Any two surfaces of revolution having a common axis will intersect in a circle or circles, perpendicular to the axis. When the axis is parallel to the plane of projection these intersections appear as straight lines. (See Fig. 101).

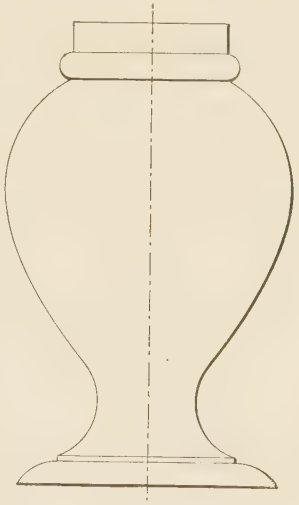


FIG. 101.—Surface of revolution.

226. Representation.—Two views of double-curved surfaces of revolution are all that are necessary, and usually one is sufficient, and that in outline. A center line on V, perpendicular to GL, is almost always drawn, and the meridian section parallel to V constitutes the V-projection. The H-projection is merely the circle of a diameter equal to the largest dimension of the generatrix, with as many other circles, visible or invisible, as may be necessary to show the various intersections of the surface. Fig. 102 shows a torus, which will fairly represent all such surfaces.

Any view of a sphere is a circle, so both projections of any sphere will be circles.

227. Problem 82.—To locate a point on any Surface of Revolution.

Let it be required to locate O on the torus in Fig. 102, assuming either projection.

Analysis.—Any plane parallel to H will cut the torus in a circle.

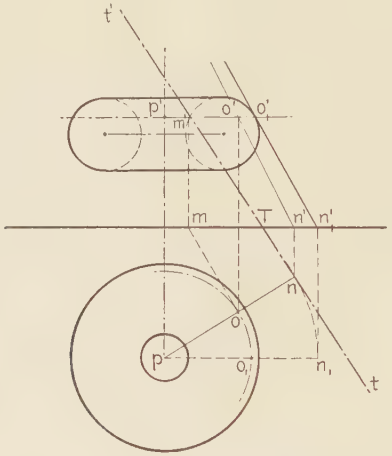


Fig. 102.

Construction, assuming the V-projection, o'.

1. Through o' pass a plane parallel to H. It will cut a circle of a radius equal to OP (o'$_1$ p').
2. Draw the H-projection of that circle, with its center at p.
3. Project o' to o on the circle.

Note.—As o' could project to two different points on the circle, it is necessary to specify "front surface" or "rear surface" to make it exact. If O were nearer the center there would be four possible locations, two on the inside surface.

Construction, assuming the H-projection, o.

1. Through o draw an arc of a circle, its radius equal to op.
2. Project o$_1$ (where the arc crosses the meridian that is parallel to V) to the given meridian curve, at o'$_1$.
3. Draw the trace of the horizontal plane through o'$_1$.
4. Project o to o' on this trace.

Note.—Two points are possible in this case, so "upper surface" and "lower surface" must be specified.

228. Tangent Planes.—As in other surfaces, a tangent plane to a double-curved surface is the locus of all tangent lines at any given point, each tangent to one of the infinite number of curves of the surface passing through that point.

The tangent plane to any surface of revolution is perpendicular to the meridian plane containing the point of tangency.

229. Problem 83.—To pass a Plane Tangent to a Sphere at a Point on the Surface.

Analysis.—Pass a plane through the point perpendicular to the radius.

Let the student make the construction.

230. To Pass a Plane Tangent to any Double-Curved Surface of Revolution at a given Point on the Surface.

Let the given point be O on the surface of the torus, Fig. 102.

Analysis.—1. Draw a tangent to the right section through the point.

2. Draw a tangent to the meridian section at the point.

3. These two tangents will determine the plane.

Construction.—1. Draw the tangent OM to the circle through O, and locate its *V*-piercing point M.

2. Revolve O to $o_1o'_1$, so that the meridian plane will be parallel to *V*.

3. Draw the tangent $o'_1n'_1$ to the meridian curve, in revolved position, and locate its *H*-piercing point N.

4. Counter-revolve N to its original position n′n, into the trace of the meridian plane through O.

5. The two tangents will now determine the tangent plane, Tt being parallel to om, and Tt′ being drawn through m′.

231. Problem 84.—To pass a plane tangent to a sphere through a given straight line.

Analysis.—Regard the sphere as being enveloped by a cylinder parallel to the given line, and pass the plane tangent to the cylinder. The plane will be tangent also to the sphere.

Construction.—Let AB (Fig. 103) be the given line, and O be the center of the sphere.

1. Pass a plane, Q, through O perpendicular to AB.

2. Find the point, P, in which AB pierces Q.

3. Revolve P and O about the *H*-trace of Q into *H* at p_1 and o_1, and draw a circle at o_1 of the same radius as that of the sphere.

4. Draw a tangent from p_1 to this circle.

5. Find its H-piercing point m (where it crosses Qq).

6. Draw the traces of T, the required tangent plane, through m and the piercing points of AB, c and d′.

Note 1.—This problem usually has two solutions, but if the line AB pierces the sphere, the problem is impossible.

Note 2.—The projections of the cylinder were not drawn, as it is merely an imaginary assistance to the analysis. It is not even necessary to imagine it, as the normal plane, Q, will have to pass through the center of the sphere, although it does not have to be perpendicular to AB. This is convenient, however, and would be necessary if the cylinder idea were carried out.

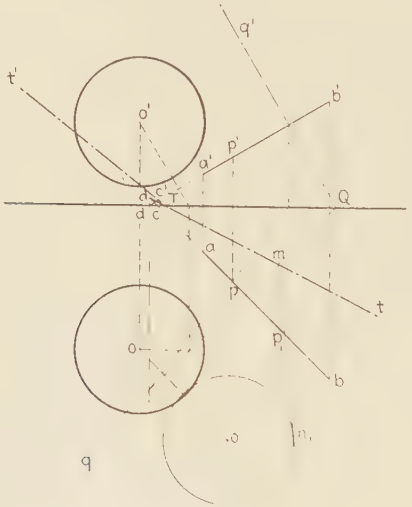


FIG. 103.

232. Problem 85.—To pass a Plane through a given Line Tangent to any Surface of Revolution.

This problem is more curious than practical.

Analysis.—First Case: If the given line intersects the axis extended.

1. Draw a cone tangent to the given surface, with the apex at the intersection of the given line and the axis.

2. Pass a plane through the given line tangent to the cone. It will be a tangent also to the surface of revolution, as required.

Analysis.—Second Case: If the given line does not intersect the axis.

1. Revolve the given line about the axis, thus generating a hyperboloid of revolution.

2. Draw a tangent line to both surfaces.

3. Revolve this line, generating a cone, to which the given line is tangent.

4. Pass a plane through the given line tangent to the cone. It will be tangent to the given surface.

Construction.—Let it be required to draw a plane T through AB, Fig. 104, tangent to the torus, whose axis is OP.

1. Revolve AB about OP, generating the hyperboloid, part of whose outline is the curve c_1' a_1'.

2. Draw the tangent, d_1' e_1', to torus and hyperboloid.

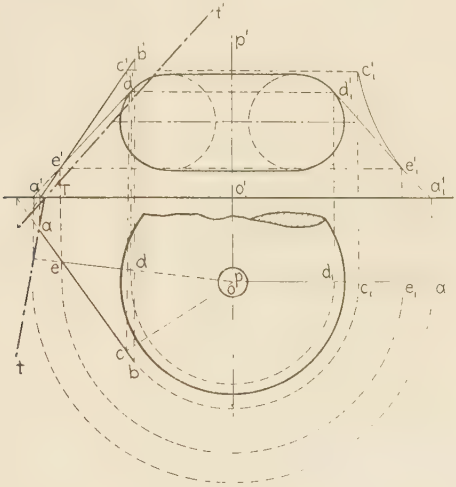


Fig. 104.

3. Revolve the meridian plane in which D and E, the points of tangency, are located, until the point E reaches the line AB. ED is then a line tangent to both surfaces intersecting AB, which is an element of the hyperboloid, and therefore tangent to the cone generated by ED.

4. Pass the plane T through AB and ED.

233. Problem 86.—To obtain the Intersection of a Plane with any Surface of Revolution.

Analysis.—The simplest curve that can be cut from a surface of revolution is the circle, which is cut by a plane perpendicular to the axis. Therefore a series of planes perpendicular to the axis will cut straight lines from the plane and circles from the given surface. The curve of intersection will pass through the

various intersections of the straight lines and circles. Fig. 105 shows both projections of the line of intersection of the plane T with a torus.

The vertices (high and low points) of the curve may be exactly located by projecting the torus and plane on a plane perpendicular to the *H*-trace of the given plane. The vertices always lie in the meridian plane which is perpendicular to the given plane.

Note 1.—Any plane will cut a circle from a sphere, because any diameter of the sphere may be considered its axis.

Note 2.—The foregoing method is often useful in obtaining the intersections of planes with **ruled** surfaces of revolution, particularly the hyperboloid.

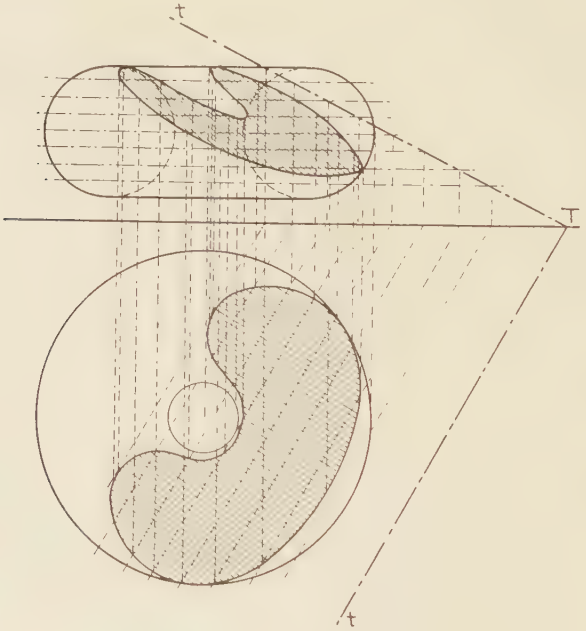


Fig. 105.

234. Development of Double-Curved Surfaces of Revolution.—Development in the strict sense is impossible, because there are no straight lines. However, in ornamental sheet metal work and the manufacture of globes, approximate surfaces are made from flat sheets.

This is done by two methods, the "**Meridian Method,**" in which

strips are cut along the meridian lines, and the "**Zone Method**" in which belts of the surface, zones, are regarded as conical surfaces. Both methods are shown in Fig. 106 and should require no explanation.

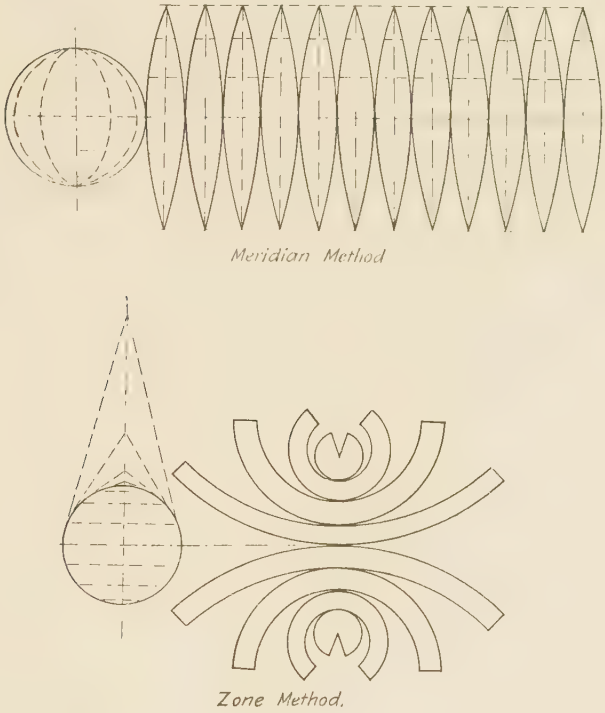


FIG. 106.—The approximate development of a sphere.

235. EXERCISES

701. Locate A($2+\frac{1}{2}$, x) on the front surface and B(4, y$-\frac{3}{4}$) on the lower surface of the 3-in. sphere, whose center is C(3, 0, 0).
702. Locate D($2\frac{3}{4}$, y$-\frac{3}{4}$) on the lower surface, and E($3\frac{3}{8}+\frac{5}{8}$, z) on the rear surface of the $1\frac{1}{2}$-in. sphere, whose center is F($3+1-1$).
703. Locate G($3\frac{1}{2}$, x$-\frac{3}{4}$) on the upper surface of the sphere in Ex. 701, and draw the traces of the plane tangent at that point.
704. Locate K($3\frac{1}{4}+\frac{1}{2}$, y) on the rear surface of the sphere in Ex. 702, and find the traces of the plane tangent at that point.
705. Sphere in Ex. 701. Find the section, and its true size, made by the plane T($1+2$) 5($1-3$).
706. Sphere in Ex. 702. Find the section, and its true size, made by the plane S($+2$)∞($-1\frac{1}{2}$).

707. Find the points in which K(1, 0−2) L(5+1¾, 0) pierces the sphere in Ex. 701.

708. Find the points in which M(2+1¾−1) N(4+½−¼) pierces the sphere in Ex. 702.

709. Develop the surface of a 1½-in. sphere by the zone method.

710. Develop the surface of a 1½-in. sphere by the meridian method.

711. Draw the projections of an oblate spheroid, its minor axis A(3+½−1¾) B(3+2½−1¾), and its major axis C(1½+1½−1¾) D(4½+1½−1¾). Locate the points E(2½+2, x) on the front and F(4, y−1¼) on the lower surface.

712. Find the traces of the plane tangent to the oblate spheroid in Ex. 711, at F(4, y−1¼).

713. Oblate spheroid in Ex. 711. Find the traces of a tangent plane containing G(4½+1−½) || to H (1+1½−½)K(0, 0−1).

714. Oblate spheroid in Ex. 711. Find the section, and its true size, made by T(1+3) 5(3−3).

715. Oblate spheroid in Ex. 711. Find where it is pierced by M(1+3−2) N(5+1−1).

716. Develop by the zone method the spheroid in Ex. 711.

717. Draw the projections of the prolate spheroid, whose major axis is O(3+¼−1½) P(3+3¼−1½), and whose minor axis is A(2+1¾−1½) B(4+1¾−1½). Locate C(2½+2½, x) on the front and D(3½, y−1) on the lower surface.

718. Spheroid in Ex. 717. Draw the traces of a plane tangent at D(3½, y−1) on the lower surface.

719. Spheroid in Ex. 717. Draw the traces of a tangent plane containing the line E(1½, 0−1) F(2½+2−½).

720. Spheroid in Ex. 717. Find the section, and its true size, made by the plane S(1+3) 4½(4½−3).

721. Spheroid in Ex. 717. Find the section, and its true size, made by the plane R(+2)∞(−3).

722. Spheroid in Ex. 717. Develop the surface by the meridian method.

723. Draw the projections of a torus, generated by a 1-in. circle revolving about M(3, 0−1¾) N(3+2−1¾), as an axis, with a center at O(2+1−1¾). Locate K(2½, x−1¼) on the inside upper surface and P(3¾+¾−y) on the rear surface.

724. Torus in Ex. 723. Draw the traces of the plane tangent at P(3¾+¾−y) on the rear surface.

725. Torus in Ex. 723. Draw the traces of a tangent plane containing A(3½+2, 0) B(5, 0−1).

726. Find the section of the torus (Ex. 723) made by the plane T(1+3) 4(3−3½). Also its true size.

727. Find the section of the torus in Ex. 723 made by the profile plane through C(3¼+1−1).

728. Find the section, and its true size, of the torus in Ex. 723, made by the plane R(+2)∞(−3).

729. Draw the projections of an ovoid, whose axis is M(3+½−1½) N(3+3−1½), whose upper radius is ½ in., and lower radius ¾ in. Locate O(3¼+⅞, x) on the front surface.

730. Find the traces of the plane tangent to the ovoid in Ex. 729 at the point O($3\frac{1}{4}+\frac{7}{8}$, x) on the rear surface.

731. Find the section, and its true size, of the ovoid in Ex. 729, made by the plane T(1+2) 5(3−3).

732. Find the plane tangent to three spheres, all resting on H, whose respective centers are A(2+1−1), B($3+\frac{1}{2}-1\frac{1}{2}$) and C($3\frac{1}{2}+\frac{5}{8}-\frac{1}{2}$).

733. Develop the surface of the ornamental sheet metal surface of revolution, shown in Fig. 107. Any convenient scale.

Note.—One strip, or meridian, will be sufficient.

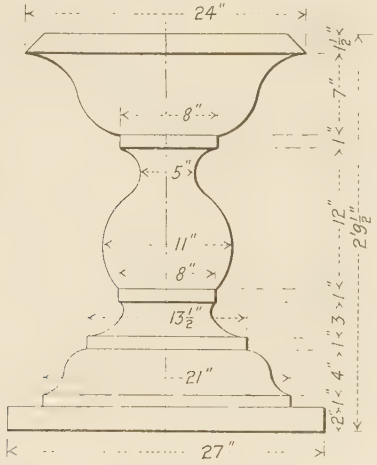


Fig. 107.—Lawn ornament.

INTERSECTIONS AND DEVELOPMENTS OF ALL SURFACES

236. General Rule for Intersections.—Pass a series of surfaces through the two intersecting surfaces, which shall cut the simplest possible lines from them. These lines, thus cut, intersect each other two and two, and by joining these points, we obtain the curve of intersection.

Most of the auxiliary surfaces used for this purpose are planes, and sometimes spheres are used, but other surfaces very seldom, never in the writer's experience.

237. The Simplest Lines Cut from Surfaces by Planes.—In selecting planes to cut two intersecting surfaces, the following table is useful, showing how to pass the proper planes to cut the simplest lines.

Surface	*Intersection*	*Kind of Plane*
Cone	Straight line	Through the apex.
Cone	Circle	Parallel to circular base.
Cylinder	Straight line	Parallel to the axis.
Cylinder	Circle	Parallel to a circular base.
Convolute	Straight line	Tangent to helical cylinder.
Convolute	Involute	Perpendicular to the axis.
Helicoid	Straight line	Through the axis.
Warped surface	Straight line	According to its generation.
Surface of revolution	Circle	Perpendicular to the axis.
Surface of revolution	Meridian curve	Through the axis.

It may be noted that planes cutting straight lines from warped surfaces are seldom easy to use because of the variety of their directions. There are cases of surfaces having a plane director where it is convenient to get straight line intersections.

It is often not possible to use the planes in the foregoing list for **both** of the intersecting surfaces, but it is always possible to use them for one. In such cases the second surface may be cut in somewhat complicated curves. It is such cases as these that make it impossible to lay down rules for all cases. The draftsman must use his judgment and instinct for the right auxiliary surfaces.

CONCENTRIC SPHERES METHOD

When two surfaces of revolution have intersecting axes the easiest method of obtaining their intersection is to use **concentric spheres.** By employing a sphere, as an auxiliary surface, with its center at the intersection of the axes, circles are cut from the two surfaces, because the intersections of two surfaces of revolution having a common axis will be circles; Article 225. A typical construction will be shown in Problem 90.

PROBLEMS IN INTERSECTIONS

A few representative problems of the many that might be given are selected for illustration.

238. Problem 87. To find the intersection between two cones. Even in so specific a problem as this, the position and character of the two cones have much to do with the choice of auxiliary planes to be used. We give two representative cases, and both could be successfully worked by using another system of auxiliary planes.

First Case.—When the cones are oblique and their bases are circles in the same plane.

Analysis.—Pass a series of planes parallel to the bases, intersecting the cones in circles. Study Fig. 108 for the solution and construction. Let the student work this by passing planes through the apices of the cones. The advantage of the latter method is that it gives definitely the outside elements of the cones that are involved in the intersection.

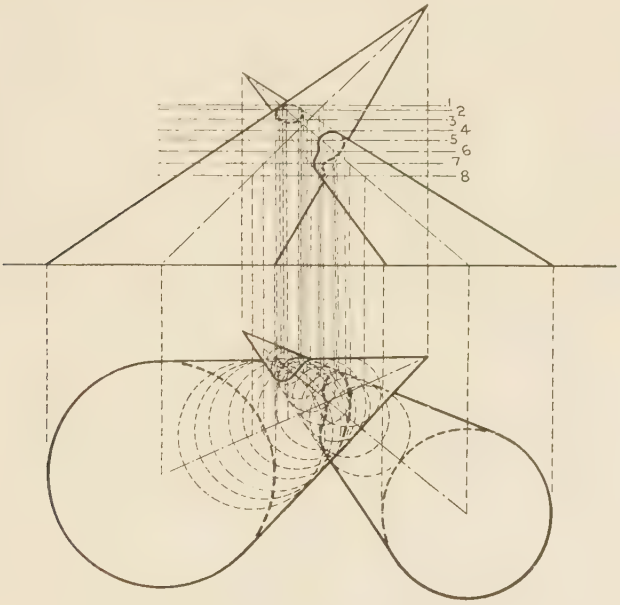


FIG. 108.—The intersection of two cones, obtained by passing planes parallel to the bases.

239. Second Case.—When the cones are right circular, and their bases are in different planes.

Analysis.—1. Pass a series of planes through the apices of the cones, cutting elements from each.

2. Connect the intersections of the elements in order.

Note.—In these and succeeding problems, it often only complicates the problem to divide each or either surface into a number of equal parts. The best way is to let the auxiliary surfaces determine which elements to draw.

Construction.—To find the intersection between the two right circular cones, whose apices are A and B, and whose bases are in P and H respectively (Fig. 109.)

1. Connect A and B with a line. This line will be contained in all the planes passing through the apices, and its piercing points will lie in all the respective traces of these planes.

2. Find the *H*-piercing point, C, of the line AB, and the *P*-piercing point, D, of the same line.

3. Through c draw any number of traces of auxiliary planes, and draw the elements cut by them from the cone B.

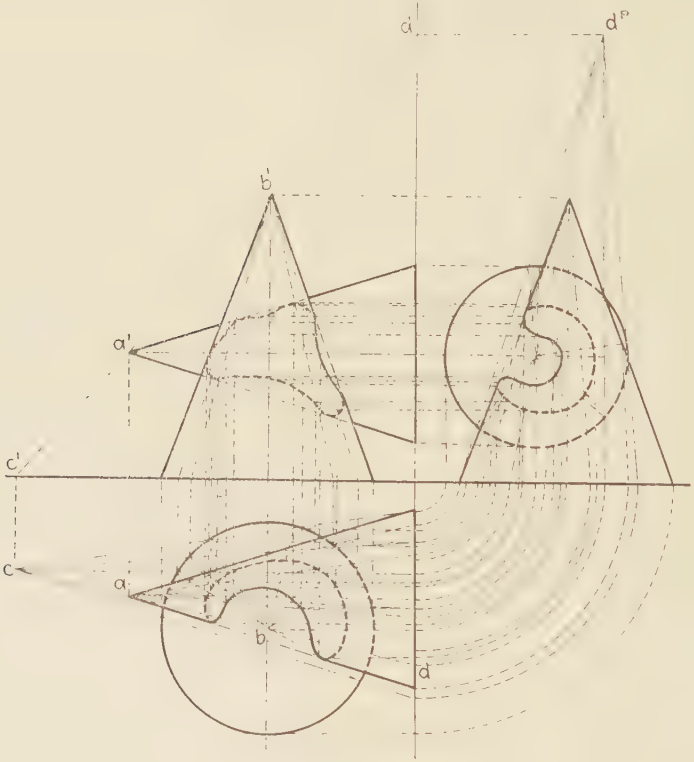


Fig. 109.—The intersection of two cones obtained by passing planes through their apices.

4. Draw the profile traces of the same auxiliary planes, and draw the elements cut by them from the cone A.

5. Locate the intersections of all the elements with the elements lying in their respective planes, obtaining the *H*-, *V*- and *P*-projections of each.

6. Draw the line of intersection through them in order, and determine the visible and invisible parts of each projection of the curve.

240. Problem 88. To find the intersection of two cylinders.

This problem, like those preceding, may have more than one means of solution, according to their relative situations and characteristics.

In the case of the cylinders in Fig. 110, two methods are available, as follows:

(A) Pass a series of planes parallel to both axes, cutting elements from both surfaces.

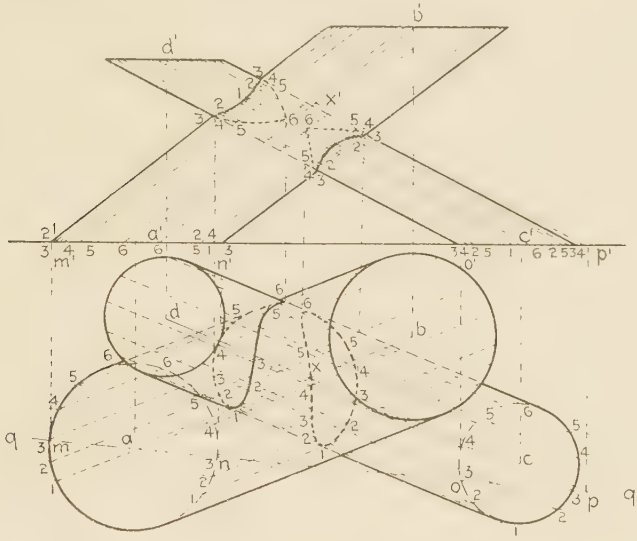


FIG. 110.—The intersection of two cylinders found by passing planes parallel to both axes.

(B) Pass a series of planes parallel to both circular bases, cutting circles from the surfaces. Method B can only be employed when the circular bases are in the same plane, as they are in this figure.

In other cases, it may be necessary to pass a series of planes parallel to one of the planes of projection; or, parallel to one axis and to the circular base of the other cylinder.

In the specimen solution, here given, planes are passed parallel to both axes.

Construction.—Let it be required to find the intersection between the cylinders in Fig. 110, whose axes are AB and CD, both having circular bases in *H*.

1. Draw the *H*-trace of a plane Q, passed through AB parallel to CD. See Problem 8.

2. This trace intersects the base circles at m, n, o and p, with V-projections at m′,n′,o′ and p′ in GL.

3. Through M, N, O, and P draw elements parallel to the respective axes.

4. These four elements lie in the same plane, and will intersect each other in four points on the curve. Locate these points.

5. Draw a series of traces parallel to qq, and repeat operations (2), (3), and (4) with each.

6. Draw a smooth curve through all of the points, taking them in order.

Note.—Do not divide either circle into any number of equal arcs. Let the planes determine the elements.

241. **Problem 89.—To find the intersection of a cone and cylinder.**

The following methods of passing planes may be employed:

1. Through the apex of the cone, parallel to the axis of the cylinder.

2. Parallel to the bases in any plane. (This method is very convenient if both surfaces have circular bases in H, V or P.)

3. Parallel to the elements of the cylinder and perpendicular to the axis of the cone, if it be a right circular cone, if the conditions are right.

Concentric Sphere Method.—If both are surfaces of revolution with axes intersecting, the process in Problem 90 is usually the best.

Auxiliary Plane Method.—There are certain conditions when the easiest and neatest solution is made by projecting both surfaces on a plane perpendicular to the axis of the cylinder. The specimen solution given here is for such a condition.

Construction.—Let AB (Fig. 111) be the axis of a right circular cone, lying on H, and CD be the axis of a circular cylinder, also lying on H.

1. Draw the projections of the cone and cylinder, drawing a sufficient number of elements of the cone.

2. Draw the H-trace (qq) of an auxiliary plane Q, perpendicular to the axis of the cylinder, and therefore perpendicular to H.

3. Project the two surfaces on Q. The cylinder will project as a circle, tangent to qq, and the elements of the cone will project as follows: A will project to a_1 on qq, as a′ is on GL, and

the points 1, 2, 3, etc., will project the same distances above qq that their respective V-projections are above GL. Why?

4. From the points on the circle, where it is crossed by the various elements, project to the various elements in their H-projections, and thence to their V-projections.

5. Draw the curve through the points in order.

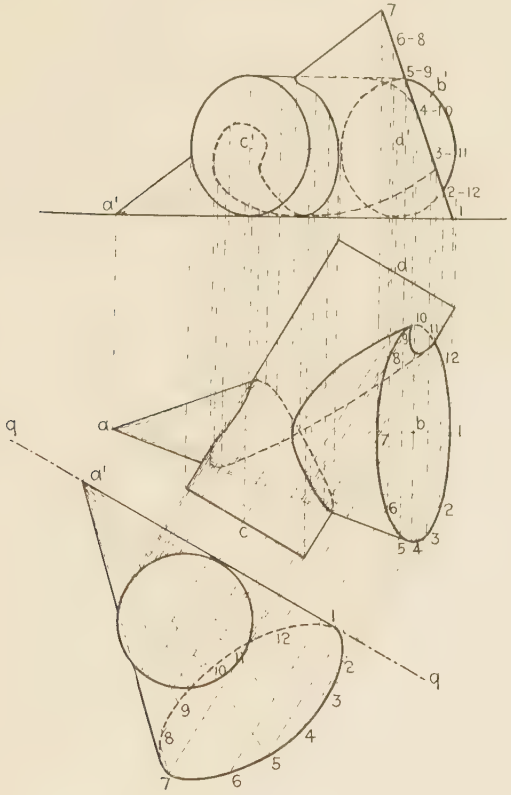


Fig. 111.

Hint.—To avoid confusion, number your points and elements, because the following out of the correct order of these points is often very difficult.

Note.—This problem can be done easily also by passing planes through the apex of the cone, parallel to the elements of the cylinder, but the method given is preferable.

242. Problem 90.—To find the intersection of any two surfaces of revolution whose axes intersect.

Analysis.—1. At the intersection of the axes as a center draw the projections of a number of concentric spheres.

2. These spheres will cut circles from the two surfaces; see Art. 237.

3. The intersections of the proper circles will yield points on the required intersection.

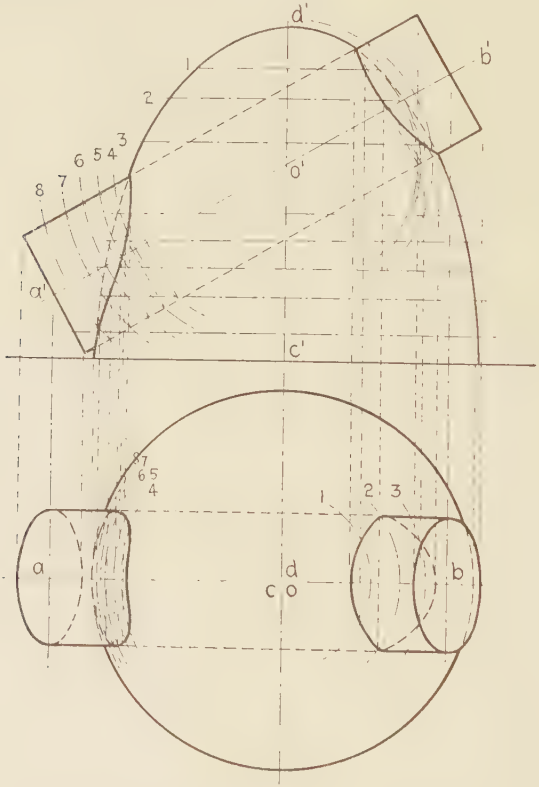


FIG. 112.—The employment of concentric spheres to determine the line of intersection between two surfaces of revolution, having intersecting axes.

Construction.—Let the circular cylinder, whose axis is AB (Fig. 112), and the ellipsoid, whose axis is CD, be the two surfaces of revolution, with axes intersecting at O.

1. With o′ as a center, draw the *V*-projections of several spheres. The *V*-projections of the circles cut by these spheres will be straight lines. Why?

2. The circles cut from the ellipsoid will project on H as circles. Draw these projections.

3. From the V-projections of the intersections of the proper circles project to the circles 1, 2, 3, etc. To do this it will not be necessary to draw the H-projections of the cylinder circles. Why?

4. Draw the two projections of the lines of intersection through these points in order.

Note 1.—The elements of the cylinder are not necessary in this solution.

Note 2.—If the position of the surfaces is not as advantageous as in this figure, project them on an auxiliary plane parallel to both axes. However, nearly all such problems are given in this position, and the difficult positions will hardly ever be encountered.

243. Discussion on Intersections and Developments of Curved Surfaces.

In attempting this kind of work a perfect knowledge of the characteristics of each surface is indispensable. One must recognize from the projections of the surface whether it is single-curved, double-curved, warped, or plane. Much choice of auxiliary surface will exist even when the surfaces are recognized, but the well-equipped student should by this time have sufficient judgment, or intuition, to choose the most suitable sections. You must not be limited to one way of working these problems, but should have all the resources available for every possible contingency, and be able to apply them to the problem in hand.

244. EXERCISES IN INTERSECTIONS AND DEVELOPMENTS

734. Find the line of intersection of the 2-in. cylinder, A, whose axis is C(3, 0−1½) D(3+2−1½), and the 1¼-in. cylinder, B, whose axis is E(1+1−1½) F(5+1−1½).
735. Develop the surface of cylinder A in Ex. 734.
736. Find the line of intersection of the 2-in. cylinder, A, whose axis is K(3, 0−1¼) L(3+2−1¼), and the 1-in. circular cylinder, B, whose axis is M(1½+½−1½) N(4½+2−1½).
737. Develop the surface of cylinder A in Ex. 736.
738. Develop the surface of cylinder B in Ex. 736.
739. Find the line of intersection of the 1¾-in. cylinder, A, whose axis is C(3, 0−1½) D(3+2−1½), and the 1½-in. circular cylinder, B, whose axis is E(2½+½−1½) F(5½+2−1½).
740. Develop the surface of cylinder A in Ex. 739.
741. Develop the surface of cylinder B in Ex. 739.

742. Find the intersection of the 2-in. cylinder, A, whose axis is G(3, 0−1¼) H(3+2−1¼) and the 1¼-in. cylinder, B, whose axis is K(1+1−1½) L(5+1−1½).
743. Develop the surface of cylinder A in Ex. 742.
744. Develop the surface of cylinder B in Ex. 742.
745. Find the intersection of the 1½-in. circular cylinder, A, whose axis is M(2, 0−1) N(4+2−1) and the 1¼-in. circular cylinder, B, whose axis is O(2+2−1¼)P(4, 0−1¼).
746. Find the intersection of the elliptical cylinders, A, whose base is a 1-in. circle in *H*, axis C(2,0−2)D(4+2−1), and B, whose base is a 1½-in. circle in *H*, and whose axis is E(2+2−1½)F(4, 0−1½).
747. Find the intersection of the 1½-in. circular cylinder, whose axis is A(2, 0−1) B(2+2−1) and the elliptical cylinder, base a 1½-in. circle in *H*, axis C(3½, 0−2) D(1+2−1).
748. Find the intersection of the 1¼-in. circular cylinder, A, whose axis is E(1+⅝−½) F(4+⅝−2), and the elliptical cylinder, B, whose axis is G(2½, 0−1) H(3+2−½), and base a 1¼-in. circle in *H*.
749. Find the intersection of two elliptical cylinders whose axes are K (1½, 0−2) L(4+2−1) and M(2+1½, 0) N(3½+½−2), and whose respective bases are a 1¼-in. circle in *H*, and a 1½-in. circle in *V*.
750. Find the intersection of the 1½-in. cylinder, whose axis is A(3, 0−1½) B(3+2−1½) and the equilateral triangular prism of 1-in. sides whose center line is C(1½+1−½) D(4+1−2).
751. Develop the surface of the cylinder in Ex. 750.
752. Develop the surface of the prism in Ex. 750.
753. Find the intersection of the 1½-in. cylinder, axis E(3, 0−1) F(3+2−1) and the equilateral triangular prism of 1-in. sides, whose center line is G(2+1½−¾) H(4+½−¾).
754. Develop the surface of the cylinder in Ex. 753.
755. Develop the surface of the prism in Ex. 753.
756. Find the intersection of the right cone, axis A (3, 0−1)B(3+2−1), base a 1¾-in. circle at A, and the 1-in. cylinder, whose axis is C(2+1−1) D(4+1−1).
757. Develop the surface of the cone in Ex. 756.
758. Develop the surface of the cylinder in Ex. 756.
759. Find the intersection of the right cone, whose axis is E(3, 0−1½) F(3+2−1½), base a 2-in. circle at E, and the 1-in. cylinder, whose axis is G(2+1−1¾) H(4+1−1¾).
760. Develop the surface of the cone in Ex. 759.
761. Develop the surface of the cylinder in Ex. 759.
762. Find the intersection of the right circular cone, tangent to *H*, whose axis is K(1, 0−2) L(3+1−2), and the circular cylinder, tangent to *H*, whose axis is M(1½+⅝−2) N(2¾+⅝−½).

Hint.—Project the surfaces on an auxiliary plane perpendicular to the axis of the cylinder.

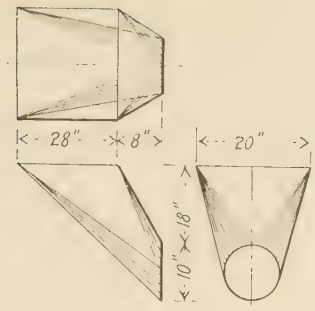
763. Find the intersection of the right cone, whose axis is A(3, 0−1½) B(3+2−1½), and base a 2-in. circle in *H*, and the 1-in. circular cylinder, whose axis is C(1½, 0−1¾) D(4+1½−1¾).

764. Find the intersection of the right cone in Ex. 763, and the $\frac{3}{4}$-in. circular cylinder, whose axis is E($3\frac{1}{2}$, $0-1\frac{3}{4}$) F($3\frac{1}{2}+2-1\frac{3}{4}$).
765. Develop the cone in Ex. 763.
766. Develop the cylinder in Ex. 763.
767. Find the intersection of two oblique cones, whose axes are E(2, $0-2$) F($3\frac{1}{2}+2\frac{1}{2}-\frac{1}{2}$), and G(4, $0-1\frac{1}{2}$) H($2\frac{1}{2}+1\frac{3}{4}-\frac{3}{4}$), with bases $1\frac{1}{2}$-in. and $1\frac{1}{4}$-in. circles at E and G.
768. Find the intersection of two oblique cones, whose axes are A(2, $0-1\frac{1}{2}$) B($3+2-1\frac{1}{2}$) and C($3\frac{1}{2}$, $0-1\frac{1}{2}$) D($2\frac{1}{2}+1\frac{1}{2}-1\frac{1}{2}$), with bases $1\frac{1}{2}$-in. and 1-in. circles at A and C.
769. Develop the surface of the cone EF in Ex. 767.
770. Develop the surface of the cone GH in Ex. 767.
771. Develop the surface of the cone AB in Ex. 768.
772. Develop the surface of the cone CD in Ex. 768.
773. Find the intersection of two right circular cones, whose axes are K(3, $0-1$) L($3+2-1$) and M($2+1-1$) N($4+1-1$), and bases are 2-in. and $1\frac{1}{4}$-in. circles at K and N.
774. Develop the surface of the cone KL in Ex. 773.
775. Develop the surface of the cone MN in Ex. 773.
776. Find the intersection of two right circular cones, whose axes are P(3, $0-1\frac{1}{2}$) O($3+2-1\frac{1}{2}$) and R(4, $0-1\frac{1}{2}$) Q($4+1\frac{1}{2}-1\frac{1}{2}$), whose bases are 2-in. circles at P and R.
777. Develop the surface of the cone PO in Ex. 776.
778. Develop the surface of the cone RQ in Ex. 776.
779. Find the intersection of the cones (right circular), whose axes are A(3, $0-1$) B($3+2-1$) and C($1\frac{1}{2}$, $0-1$) D($4+\frac{5}{8}-1$). The base of the first is a 2-in. circle at A, and the second cone is tangent to *H*.
780. Develop the cone AB in Ex. 779.
781. Find the intersection of the right circular cones, whose axes are E(3, $0-2$) F($3+2\frac{1}{2}-2$) and G($3+\frac{1}{2}-2$) H($4\frac{1}{2}+1\frac{1}{2}-2$), and whose bases are 2-in. and 1-in. circles at E and G.
782. Develop the surface of both cones in Ex. 781.
783. Make frusta of the two cones in Ex. 781 by passing planes perpendicular to the axes, 1 in. from F and $\frac{1}{2}$ in. from the apex H. Develop the surfaces of both (Coffee Pot).
784. Find the intersection of the oblique cone, whose axis is K($2+2$, 0) L($3\frac{1}{2}+1-2$), base a $1\frac{1}{2}$-in. circle at K, and the right cone M(3, $0-1$) N($3+2\frac{1}{2}-1$), whose base is a $1\frac{3}{4}$-in. circle at M.
785. Find the intersection of two oblique cones, having the same base, a 2-in. circle at O(3, $0-1\frac{1}{2}$), and their vertices at P($2+2-1$) and Q($4+1\frac{1}{2}-1$).
786. Find the intersection of the oblique cone, whose axis is A(2, $0-2$) B($4+2-1$), and base a 1-in. circle at A, and the elliptical cylinder, whose axis is C($3\frac{1}{2}$, $0-1\frac{1}{2}$) D($2\frac{1}{2}+2-1$), and base a $1\frac{1}{2}$-in. circle at C.
787. Develop the cone in Ex. 786.
788. Develop the cylinder in Ex. 786.
789. Find the intersection of the right circular cone, whose axis is E(3, $0-1\frac{1}{2}$) F($3+2-1\frac{1}{2}$), and base a 2-in. circle at E, and the elliptical cylinder, whose axis is G($2+2$, 0) H($3+1-2\frac{1}{2}$), base a 1-in. circle at G.

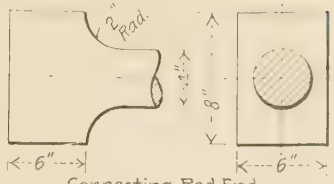
790. Develop the surface of the cone in Ex. 789.
791. Develop the surface of the cylinder in Ex. 789.
792. Find the intersection of the right circular cone, axis K(3, 0−1½) L(3+2−1½), and 2-in. base, and a triangular prism of 1-in. sides, whose center line is M(2+1−½) N(4+1−2).
793. Develop the surface of the cone in Ex. 792.
794. Develop the surface of the prism in Ex. 792.
795. Find the intersection of the oblique cylinder, whose axis is A(2, 0−2) B(4+3−½), base a 1¼-in. circle in *H*, and the 2-in. sphere, whose center is C(2½+1−1½).
796. Find the intersection of the 3-in. sphere, center at D(3+1½−2) and a circular cylinder of 1½-in. diameter, whose axis is ¾ in. from D, parallel to *H*, and inclined 30° to *V*.
797. Find the intersection of the 2-in. sphere, whose center is M(3+1½−1½) and the 1-in. circular cylinder, whose axis is N(3+2−2)O(3, 0−2).
798. Develop the surface of the cylinder in Ex. 797.
799. Develop the surface of the sphere in Ex. 797 by the zone method.
800. A torus is generated by a 1¼-in. circle, center at A(2+1−1¾), rotating about B(3, 0−1¾) C(3+3−1¾). Find its intersection with the 1¼-in. circular cylinder, whose axis is D(2, 0−1¾) E(2+2−1¾).
801. Develop the surface of the cylinder in Ex. 800.
802. Torus in Ex. 800. Find its intersection with the 1¼-in. circular cylinder, whose axis is F(2½, 0−1¾) G(4+2−1¾). (Concentric sphere method.)
803. Develop the surface of the cylinder in Ex. 802.
804. Find the intersection of an oblique cylinder, whose axis is K(2, 0−2) L(4+3−1), base a 1-in. circle in *H*, and the oblate spheroid, whose major axis is M(1½+1−1½) N(4½+1−1½), and whose minor axis is O(3, 0−1½) P(3+2−1½).
805. Find the intersection of the right circular cone, of a 2-in. base, whose axis is A(2, 0−2) B(2+2½−2), and a 2½-in. sphere, whose center is C(3+1¼−1¼).
806. Find the intersection of an oblique cone, axis D(2, 0−2) E(3½+2½−1), base a 2-in. circle in *H*, and the 2-in. sphere, whose center is F(3+1¼−1¼).
807. Torus in Ex. 800. Find its intersection with the cone in Ex. 805.
808. Torus in Ex. 800. Find its intersection with the cone in Ex. 806.
809. Develop the surface of the cone in Ex. 805, 806, 807 or 808.

PRACTICAL EXERCISES IN VARIOUS SURFACES

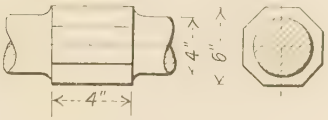
810. Lay out the patterns for the concrete-mixer hopper, shown in Fig. 113, to scale. Two spaces.
811. Find the line of intersection of the plane and curved surfaces of the connecting rod end, shown in Fig. 113. Draw three projections to scale, one space.
812. Find the lines of intersection of the various surfaces of the octagonal tie rod, or turnbuckle, shown in Fig. 113, to scale. One space.
813. Draw the projections of a 2-in. hexagonal nut, having a 45° conical



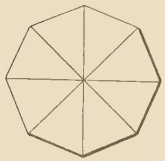
Hopper of Concrete Mixer.

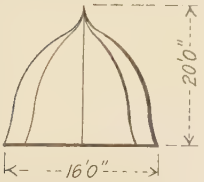


Connecting Rod End.

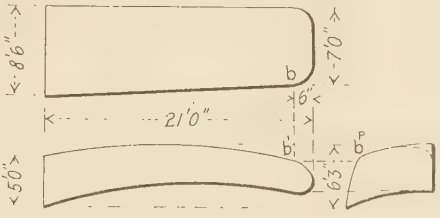


Octagonal Tie Rod.

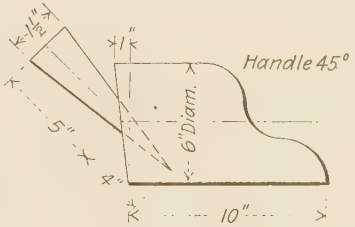




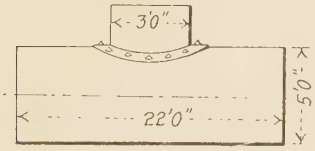
Octagonal Dome.



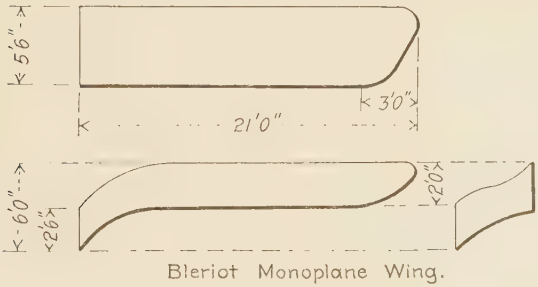
Fairchild Monoplane Wing.



Sugar Scoop.



Oil Tank



Bleriot Monoplane Wing.

FIG. 113.—Practical exercises involving various surfaces.

chamfer, and determine the exact line of intersection. Full size, one space.

814. Draw the projections of an octagonal dome, like the one shown in Fig. 113, to scale, and lay out the pattern for one strip of the surface. One space.

815. Find the line of intersection in the head of the oil tank shown in Fig. 113. Make the flange, at the seam, 5 in. wide, and draw the projections of 20 equally spaced cone-head rivets. Develop one sheet (with the hole for the head) of the body of the tank. Do not draw the entire length of the body. Any scale, two spaces.

816. Lay out the patterns to scale for the sugar-scoop, shown in Fig. 113. Two spaces.

817. Develop the surface, to scale, of the wing of the Fairchild monoplane, shown in Fig. 113. What is the character of the surface? Two spaces.

818. Develop the surface, to scale, of the wing of the Bleriot monoplane, shown in Fig. 113. What is the character of the surface? Two spaces.

819. A grain elevator has a 2-ft. square chute running downward at 5° from the perpendicular. It is proposed to run an off-chute at the bottom at 45° from the perpendicular, 12 in. square. Design the transition piece between the two chutes. Lay out its pattern to scale. Two spaces.

820. Lay out the patterns, to scale, of the hopper for a stone crusher, shown in Fig. 114. Two spaces. Note that two surfaces are conoidal.

821. Lay out the patterns, to scale, of the dust pan, shown in Fig. 114. Two spaces.

822. Draw the line of intersection, and lay out the pattern for the escape pipe joint, shown in Fig. 114. One space.

823. Design a steamship ventilator like the one shown in Fig. 114. Lay out the patterns to scale. Two spaces.

824. Find the line of intersection of the ventilator flue and its escape pipe, shown in Fig. 114. Lay out the patterns to scale. Two spaces.

825. Find the line of intersection of the reducing hood (warped cone) and its escape pipe, shown in Fig. 114. Lay out the patterns to scale. Two spaces.

826. Lay out the patterns to scale for the transition piece for the fan connection shown in Fig. 115. Two spaces.

827. Grain Hopper (Fig. 115). Lay out the patterns to scale for the grain hopper, noticing that two of the sides are hyperbolic paraboloids, and not planes. Two spaces.

828. Measuring Hopper for Automatic Filling (Fig. 115). Lay out the patterns for this conoid surface to scale. Two spaces.

829. Draft Pipes for Ventilator (Fig. 115). All pipes are 4 in., except the center one, which is 8 in. Find all the lines of intersection and lay out the patterns for the various pieces without duplication. Two spaces.

830. Bucket for Chain Conveyor (Fig. 115). Lay out the pattern to scale. One space.

Note.—This surface is not a right circular cone; what is it?

831. Motor Boat Funnel (Fig. 115). The seam between the two cones is in a plane at right angles to the center line of the larger cone. Lay out the patterns to scale. One space.

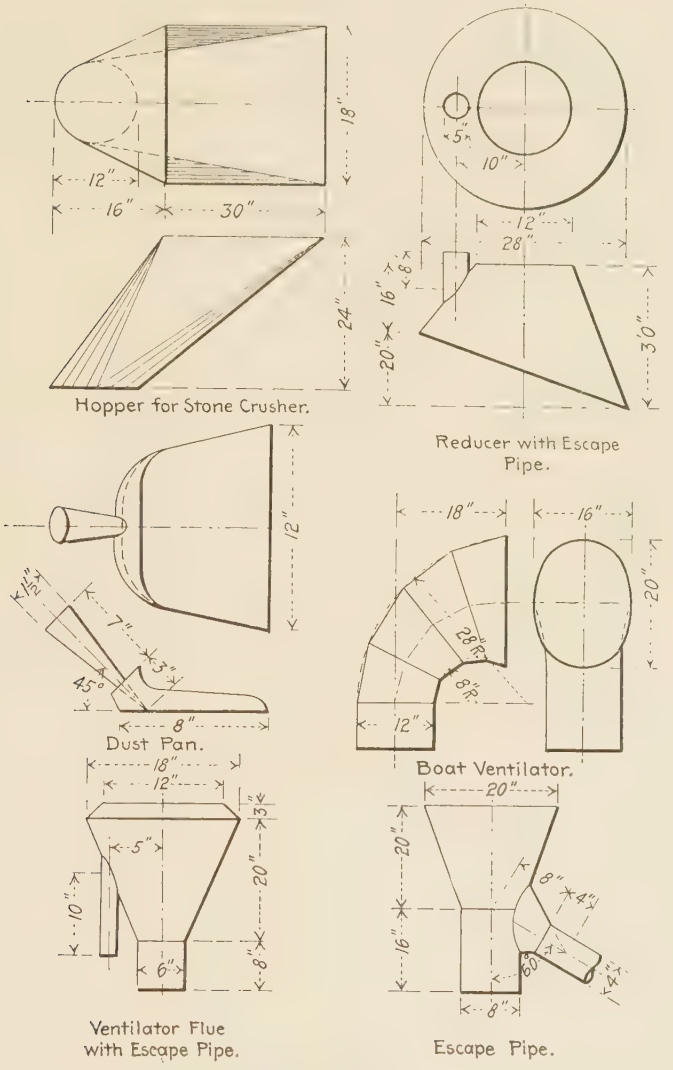


Fig. 114.—Practical exercises involving various surfaces.

832. Coffee Pot with Cylindrical Spout (Fig. 115). Find the line of intersection between body and spout, and lay out the patterns to scale. Two spaces.

833. Kettle with Conical Spout (Fig. 115). Find the line of intersection between body and spout, and lay out the patterns to scale. Two spaces.

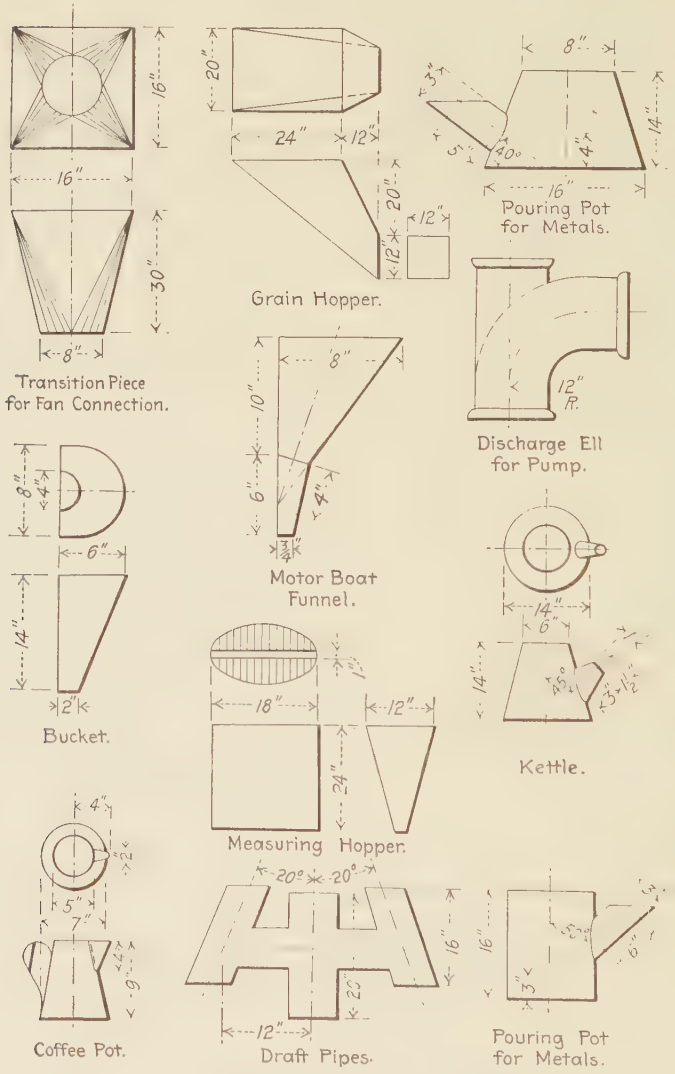
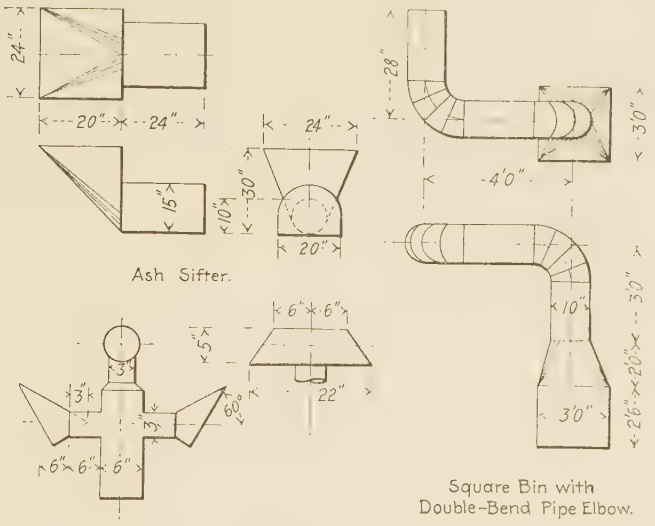


Fig. 115.—Practical exercises involving various surfaces.

834. Cylindrical Pouring Pot for Metals (Fig. 115). Find the intersection of spout and body, and lay out the patterns to scale. Two spaces.

835. Conical Pouring Pot (Fig. 115). Find the line of intersection of body and spout, and lay out the patterns to scale. Two spaces.

836. The center line of the discharge ell of the pump connection, shown in Fig. 115, makes a bend through 90° on a 12-in. radius. The outside diameters are everywhere 8 in., and the walls are $\frac{3}{4}$ in. thick.
(a) Find the line of intersection on the outside.
(b) Find the line of intersection on the inside.
(c) What is the surface of the bent pipe?



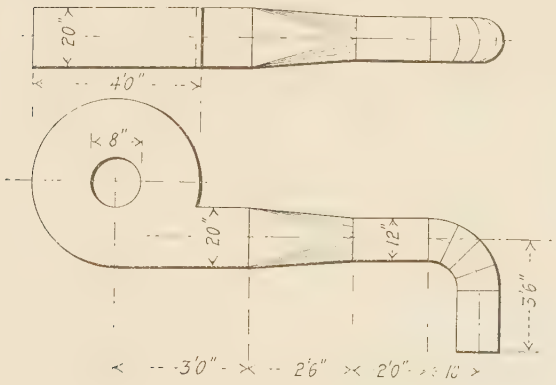


FIG. 116.—Practical exercises in surfaces.

EXCERCISES TO OCCUPY AN ENTIRE SHEET.

837. Make full drawings, lay out all the patterns, with dimensions in full, listing each piece, of the Ash Sifter, shown in Fig. 116. To scale.

838. Make full drawings to scale, lay out all the patterns, with dimensions in full, listing each piece, of the Square Bin, with Double-bend Pipe Elbow, shown in Fig. 116.

839. Make full drawings to scale, lay out all the patterns, with dimensions in full, listing each piece, of the Casing and Connections for the Blower, shown in Fig. 116.

840. Make full drawings to scale, lay out the patterns, with dimensions in full, listing each piece, of the Passenger Car Ventilator, shown in Fig. 116.

841. Make complete drawings to scale of the Air Draft Box with Draft and Blast Pipes, as shown in Fig. 117. Lay out all the patterns with dimensions and lists.

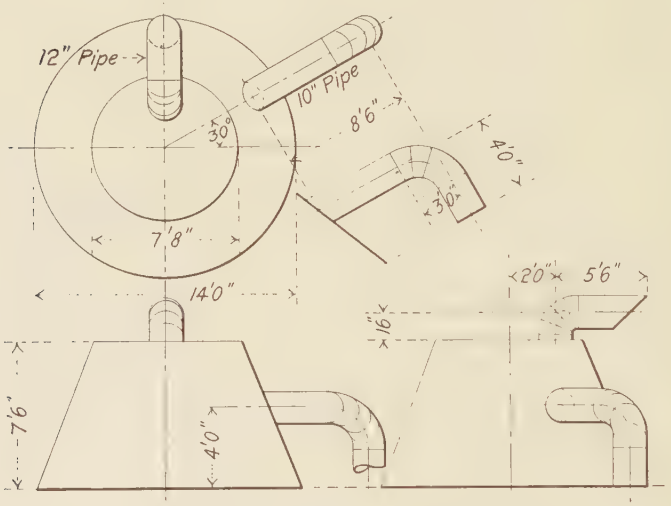


Fig. 117.—Air draft box with draft and blast pipes.

CHAPTER IV

PERSPECTIVE AND ISOMETRIC PROJECTION

PERSPECTIVE PROJECTION

245. Discussion.—Early in this work (Art. 1–2) Perspective Projection was defined and its variance from Orthographic Projection was discussed. It was shown that the resemblance of a **picture** to the **image** formed on the retina of the eye of the observer comes from the fact that, **as an object recedes, it "looks" smaller.** This is so, owing to the diminishing angle between the limiting rays, as illustrated in Fig. 1, p. 2. This gradual diminution of an object in recession must ultimately result in its **vanishing,** and this vanishing of receding objects is the basis of the science of perspective. From it we get our idea of distance in a picture, and the proper treatment of it by the artist or draftsman is what gives the picture its "life-like" appearance.

Standing between the tracks of a railroad (parallel lines), and looking in their direction, we **seem** to see them gradually approach each other and finally to meet at a distant point on the level of the eye. **All horizontal lines seem** to vanish at this level, and the fact that this line is called the **Horizon** shows the connection. The horizon is an imaginary line, of course.

From this the first rule of perspective is obtained, as follows:

All parallel horizontal lines meet one another (that is, they vanish) **at the horizon.** All horizontal lines terminate at the horizon in a picture, except those that are parallel to the plane of the picture, called the **Picture Plane.** These lines are parallel to the horizon in the picture.

All parallel lines in space are converging lines in the perspective, and their vanishing points are located on the picture plane. The location of the vanishing points of the various **systems** of parallel lines is the important operation in perspective projection, and, having located them, it is quite easy to finish the picture.

All lines parallel to the picture plane vanish at infinity; that is, they will be drawn in their true space angles in the picture. Hence, **all vertical lines are drawn vertically in the picture.**

VANISHING POINTS

246. All the parallel lines in any system vanish in one point, be they all in the same or different planes. Therefore, if the eye is turned in the direction of the lines in any system, the line of sight will meet them all in the vanishing point. Hence, to locate on the picture plane the vanishing point for any system, look in the direction of the lines of that system, holding the picture plane in its proper position, and the **vanishing point** of the system will be the point in which the line of sight pierces the picture plane.

247. The Construction of a Perspective from an Orthographic Projection.

Fig. 118 shows a set of steps in third angle projection, somewhat lower than the horizon. Do not confuse the horizon with

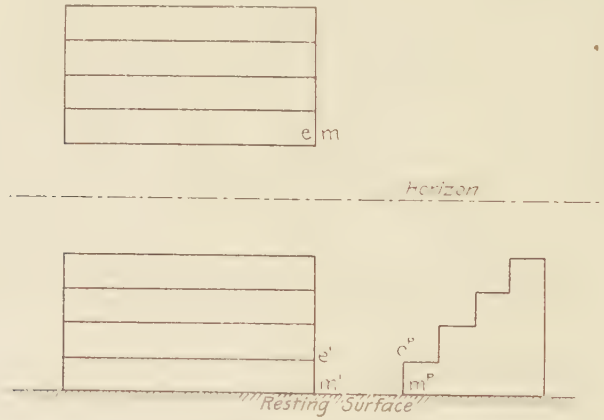


FIG. 118.—Orthographic projection of an object.

GL, as they are not used in precisely the same manner. Notice that this object has nothing but vertical and horizontal lines.

There are two ways of placing the object for a perspective, called respectively **One-Point** and **Two-Point** perspective. In the one-point, called so because there is but one vanishing point for the principal horizontal lines, the object is placed so that one system of its horizontal lines is parallel to the picture plane; that is, in the relation of the object in Fig. 118 to *V*. This style is usually employed for interiors of buildings, stage scenery, etc.

In the two-point perspective the object is placed at an angle to the picture plane; that is, "cornerwise." This is the way most objects appear to the casual observer, and is the style employed in drawing exteriors of buildings, machine parts, articles to be shown in catalogues, etc. Two-point perspective will be employed in this construction.

First: Draw the line *GL*, as in Fig. 119. It is the *H*-trace of the picture plane. Place *GL* well up on the drawing, so as to allow plenty of room for the picture.

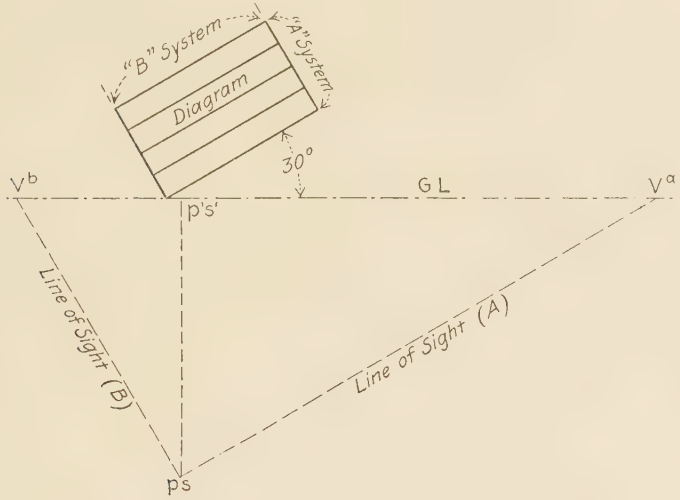


Fig. 119.—The placing of the object and locating the vanishing points.

Second: Draw the *H*-projection (Diagram) of the object at any angle, say 30°, from *GL*.

Third: Locate the **Point of Sight** (PS) at some convenient point, preferably in front of the center of the object and as far away as space will allow. If the point of sight is too near the object, the picture is distorted.

Fourth: Group the parallel lines of the object into two systems, A and B.

Fifth: Draw parallels from the point of sight to the two systems. They will pierce the picture plane at the vanishing points. Thus V^a and V^b will be the respective vanishing points for the two systems.

The next step is shown in Fig. 120. Draw lines from ps to the

various points, a, b, c, etc., of the object. These are lines of sight and pierce the picture plane at a_1, b_1, c_1, etc., on *GL*.

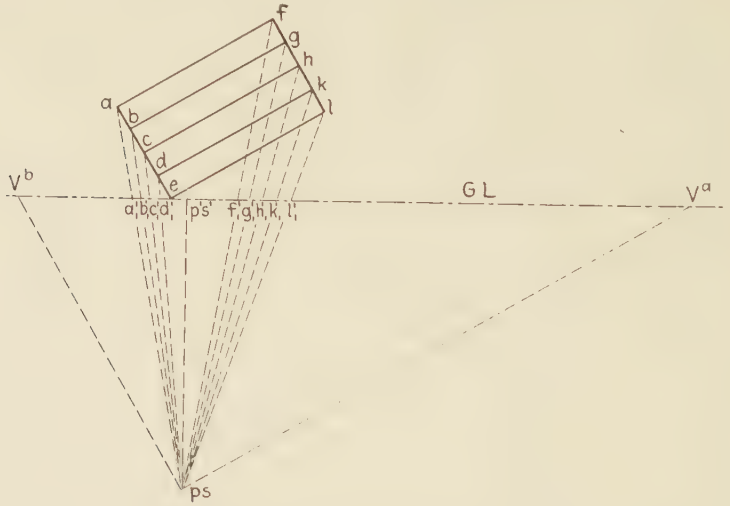


FIG 120.—Drawing the rays from the points of the object, and locating the points in which the rays pierce the picture plane.

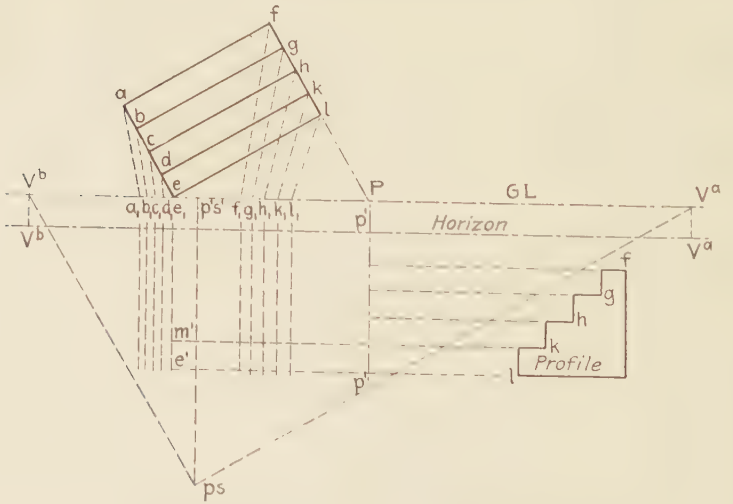


FIG. 121.—Starting the picture, drawing the horizon, etc.

The first step in drawing the picture is now shown in Fig. 121. Draw the **Horizon line,** as shown, at any convenient place, where the space is clear, above or below the point of sight. Project

the two vanishing points to the horizon. Project the points a_1, b_1, c_1, etc., found in Fig. 120, indefinitely to the space below the horizon, because in this case the object is entirely below the horizon (see Fig. 118).

All points of the object that are **in the picture plane,** are projected on the picture plane in their true distance above or below the horizon.

Hence the points E and M in this object are immediately located at e′ and m′, their true distance below the plane of the horizon.

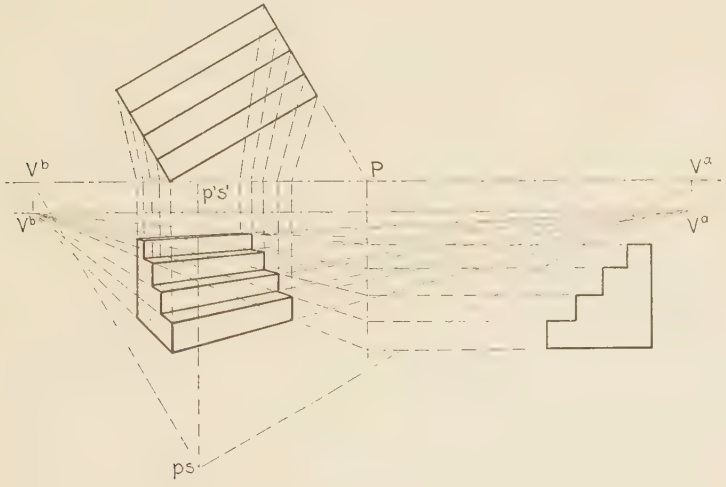


FIG. 122.—The picture completed.

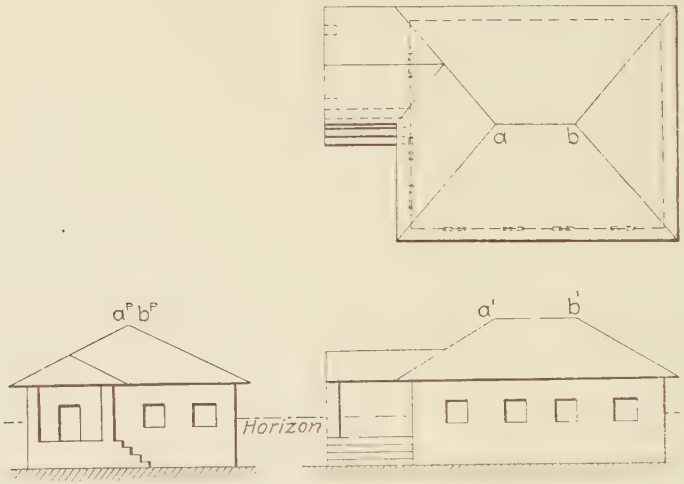
These points alone cannot determine all of the points in the picture. The points, F, G, H, K and L, all lie in one plane and are intersections of lines of both A and B systems, and verticals as well. Therefore, if we obtain the picture plane trace of the plane, P, of these points, lines of both systems may be drawn through the points thus determined.

On either side of the figure draw the profile view of the steps, in its true relation to the plane of the horizon. Project the points f, g, h, k and l to the trace p′p′. Fig. 122 shows these points connected with Vb, and intersecting the projectors in the correct points on the picture plane. Connecting these points with Va will be sufficient to determine the lines of the A-system. A little study of Fig. 122 will show how this is effected.

248. The Construction of an Object having Oblique Lines.

Such a construction as the picture of the bungalow shown in Fig. 123, in orthographic projection, involves **no new principles,** merely an elaboration of the principles already used in the previous construction. For instance, to locate on the picture plane points A and B of the ridge:

1. Extend AB in the diagram in Fig. 124, until it pierces the picture plane at C. An examination of the projections of AB



Projections of a Bungalow.

FIG. 123.

will show that it is parallel to H and a certain distance above the horizon, and belongs to the A-system of parallels.

2. Having located the point c′ in the foregoing operation, connect it with Va. The points a′ and b′ are located on this line by projecting the piercing points a_1' and b_1' of the lines of sight. All other horizontal lines are treated in the same manner, and the oblique lines are drawn from point to point on the horizontals.

249. EXERCISES IN PERSPECTIVE

In the following exercises it is advisable to draw the projections complete before starting the perspective. Place the object for each perspective "cornerwise," as in the two examples just given. No dimensions are necessary to be shown in these pictures, and invisible lines should not be drawn. The pictures should be drawn as large as space will permit, but care should

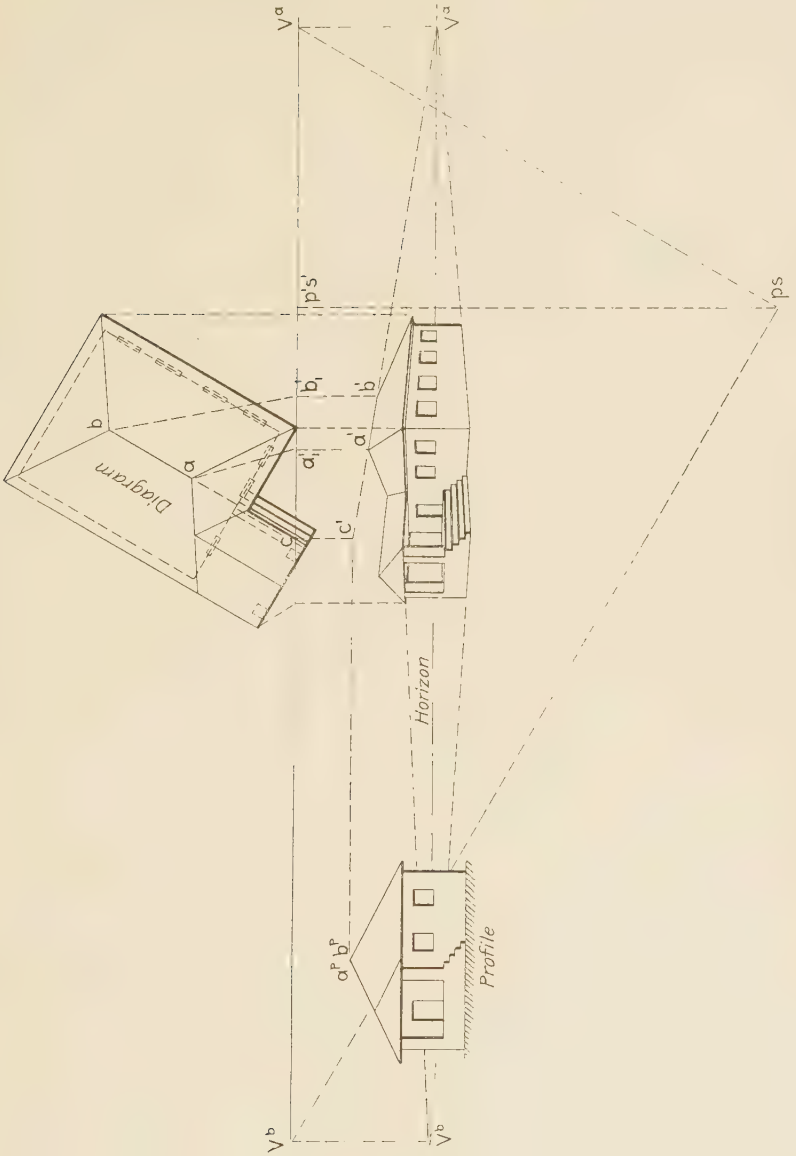


FIG. 124. Picture of the bungalow.

be taken to have the point of sight far enough from the picture plane (in proportion to the size of the object) to avoid distortion. Place all small objects below the horizon.

842. Draw the perspective of a $1\frac{1}{2}$-in. cube.

843. Draw the perspective of the frustum of a square pyramid, lower base $1\frac{1}{2}$ in., upper base $\frac{3}{4}$ in., altitude (base to base) 1 in.

844. Draw the perspective of a wooden box, to scale, 20 in. long, 12 in. wide, 6 in. deep, with walls 1 in. thick, without top.

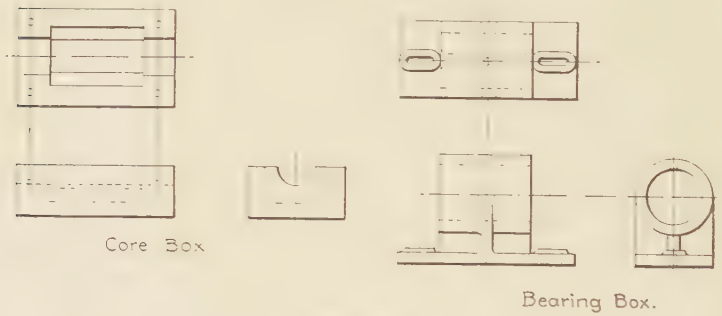


Fig. 125

Fig. 126

845. Draw a cross made of five 1-in. cubes (like Fig. 67), surmounting a 3-in. cube. Draw the projections and perspective to scale, with the horizon at the height of the center of the cross.

846. Place a $1\frac{1}{4}$-in. equilateral triangular prism, $1\frac{1}{4}$ in. long on a 1-in. cube, resembling a house with gable roof. Make the drawing and perspective.

847. Place a $1\frac{1}{4}$-in. square pyramid, $\frac{1}{2}$-in. altitude, on a 1-in. cube. Make the drawings and perspective.

848. Make the perspective of the core box for octagonal core shown in Fig. 125.

849. Make the perspective of the bearing box shown in Fig. 125.

850. Make the perspective of the house shown in Fig. 126. Locate the horizon about 6 ft. above the ground line.

ISOMETRIC PROJECTION

350. Discussion.—A convenient and very satisfactory mode of making working drawings on a single plane is **Isometric Projection,** sometimes called **Practical Perspective.** Working drawings, as usually made, require two, and often three or more, views to present all the necessary details and measurements, but, as they are made with the eye at an infinite distance, they often fail to convey to the workman a correct idea of the actual appearance of the object.

By making the drawing in Isometric Projection, we imitate Perspective in placing the object so as to present to the eye the appearance of a picture. Yet it is drawn so as to permit many of its dimensions being made to scale. But, as **distance is not taken into account,** the objects have a more or less distorted appearance when drawn in this projection, but the likeness is sufficient for its purpose.

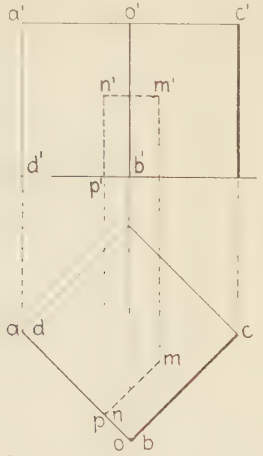
The advantage of Isometric Projection over Perspective is that its construction is very much easier and more rapid. It is not as accurate in its representation as perspective, but it has the advantage of being made to scale, as it actually is orthographic projection on a plane placed so as to show three dimensions.

In certain lines of work, such as the piping and wiring of buildings, an isometric layout is very desirable. On such a layout all necessary dimensions can be shown, and the contractor and workmen are able to make their estimates and constructions without the confusing, and often uncertain, labor of using a multitude of blue prints. The use of the isometric layout for such purposes is growing into general adoption.

251. Definition.—Conceive a cube placed as in Fig. 127, presenting its corner to the front. Let this cube be projected on a plane parallel to the plane of the points A, B and C. The three edges, AO, BO, and CO will be at equal angles to the plane of projection, and will be projected in equal lengths and

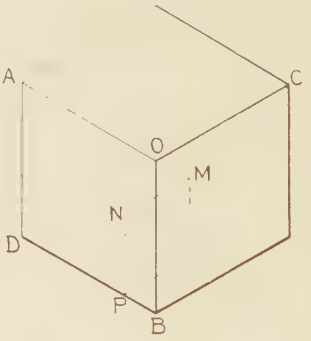
at equal angles (120°) to each other. (Prove this.) The vertical edge, OB, will be projected as a vertical line, while the horizontal edges, AO and CO, will be projected as lines making 30° with the horizontal. The complete projection of the cube will be as shown in Fig. 128, except that this figure is **actually** the projection of a larger cube, although its edges measure the same as in Fig. 127.

252. Isometric Lines.—The lines AO, BO and CO will be projected in equal lengths on the isometric plane, because they



Orthographic Projection of a Cube.

FIG. 127.



Isometric Projection of the same Cube.

FIG. 128.

are equal to each other in fact; hence the name "Isometric," which means equal measurement. These three lines (or any lines parallel to them) are called variously **directrices, coördinate axes** and **isometric lines.** We shall call them isometric lines.

Rule.—All vertical (isometric) lines are drawn vertically (as in Perspective), and all horizontal isometric lines are drawn **30 from the horizontal,** in one direction or the other, according to which system of parallel lines they are members.

Any object may be so placed as to give an isometric view by bringing its center line into a position similar to this.

253. Isometric Scale.—Although the isometric lines are projected in equal lengths **to one another,** their projected lengths are not as long as their true lengths, but, .81647 of the true

length. In Figs. 127 and 128, the cube shown in orthographic and isometric projection is given edges of equal length in both figures. Strictly speaking, this should not be done, but the lines in Fig. 128 should all be .81647 of the length they are drawn. Practically, however, no attention is paid to this foreshortening, because nearly all drawings are made to a reduced scale, so that it is not necessary to have an agreement in scales between the two. The great practical feature of this is that dimensions may be laid off on isometric lines, using any convenient scale. This means that isometric drawings are made on a slightly larger (22%) scale than the orthographic drawings. The following rules will be found useful in laying out isometric drawings.

254. Rules.—1. All vertical lines in the object are vertical in the drawing, and are drawn in their true length.

2. All horizontal lines, parallel to either **Isometric Horizontal,** will be drawn in their true length, inclined 30° to the horizontal.

3. All other lines will be foreshortened or lengthened.

4. Any point of an object may be located isometrically by means of **isometric ordinates** from known lines.

5. All straight lines may be located isometrically by the isometric projections of two of their points.

6. All curved lines may be located isometrically by the isometric projections of a sufficient number of their points.

255. Problems in Isometric Projection.

Problem 91.—To Draw a Cube.

Note.—All lines in the cube are Isometric lines.

Construction.—Let the projections of the cube be those shown in Fig. 127.

1. Draw the vertical line BO (Fig. 128), equal in length to o′b′ (Fig. 127).

2. From O draw two 30° lines, OA and OC, equal in length to BO.

3. Complete the three parallelograms determined by these three lines in pairs.

The result will be the isometric drawing of the given cube, as shown in Fig. 128.

256. Problem 92.—To Locate a Point.

Construction.—Let the point M inside the cube (Fig. 127) be given; to find its location inside the cube drawn in Fig. 128.

1 Assume the projections m,m′ in Fig. 127.

2. Draw the line mn, m′n′ parallel to OC, piercing the face at nn′.

3. From N draw the line np, n′p′, parallel to OB, until it intersects the edge BD.

These lines, MN and NP, are isometric ordinates, being parallel to isometric lines.

4. Lay off BP on BD (Fig. 128) equal to bp (Fig. 127).

5. Lay off PN parallel to BO (Fig. 128) equal top′n′ (Fig. 127).

6. Lay off NM parallel to OC (Fig. 128) equal to nm (Fig. 127).

M will be the required point inside of the cube. Any point in space can thus be located on an isometric drawing.

257. Problem 93.—To draw an exact circle.

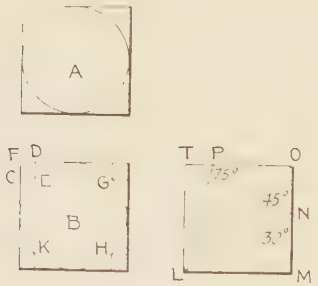


Fig. 129.

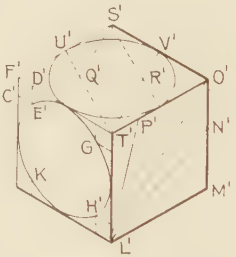


Fig. 129a.

For this and the two following problems, let us use the cube, shown in orthographic projection in Fig. 129, and in isometric projection in Fig. 129*a*.

Construction.—1. Inscribe the circle in the square B of the cube in Fig. 129.

2. In each corner of the square construct a small square with one corner in the circle at E, G, H and K; example CDEF.

3. Draw these small squares isometrically in Fig. 129*a*, with corners at E′, G′, H′, and K′; example C′ D′ E′ F′.

4. Through E′, G′, H′, and K′ draw a smooth ellipse tangent to the four sides of the square at their middle points.

258. Problem 94.—To draw an approximate circle.

This is the system almost exclusively used in actual drafting, because it can be drawn with the compasses.

Construction.—Let the circle be drawn in face A, the upper horizontal face, of the cube in Fig. 129.

1. In the upper parallelogram, F′T′O′S′ (Fig. 129*a*,) draw lines from S′ and T′ to the centers of the opposite sides, intersecting at Q′ and R′.

2. From T′ as a center, with radius T′U′ draw the arc U′V′, and do the same from S′ as a center.

3. From Q′ and R′, as centers, draw arcs with Q′U′ and R′V′ as radii, completing the curve.

259. Problem 95.—To draw lines making any required angles with any given line.

Construction.—Let it be required to draw lines making 30°, 45° and 75° with LM, on the profile face of the cube.

1. Draw LN, LO and LP in the projection of the cube (Fig. 129).

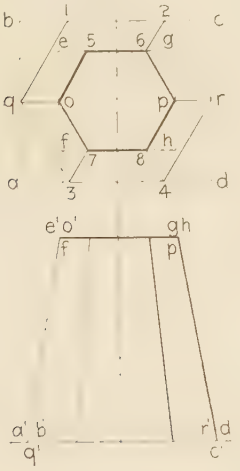


Fig. 130.

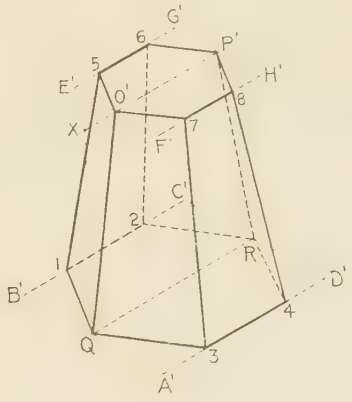


Fig. 130*a*.

2. From M′ lay off the distance MN, on M′O′, and draw L′N′. This will be the 30° line.

3. From M′ lay off the distance MO, to the point O′, and draw L′O′. This will be the 45° line.

4. From T′ lay off the distance TP, on T′O′, and draw L′P′. This will be the 75° line.

Note.—If the angles were laid off at M instead of L, they would work out differently. For instance, the angles at L′ and M′ each represent 90°, yet one is actually 60° and the other 120°, as they are drawn on the paper.

260. Problem 96.—To draw a frustum of a hexagonal pyramid.

Construction.—Let it be required to draw the frustum, whose projections are shown in Fig. 130.

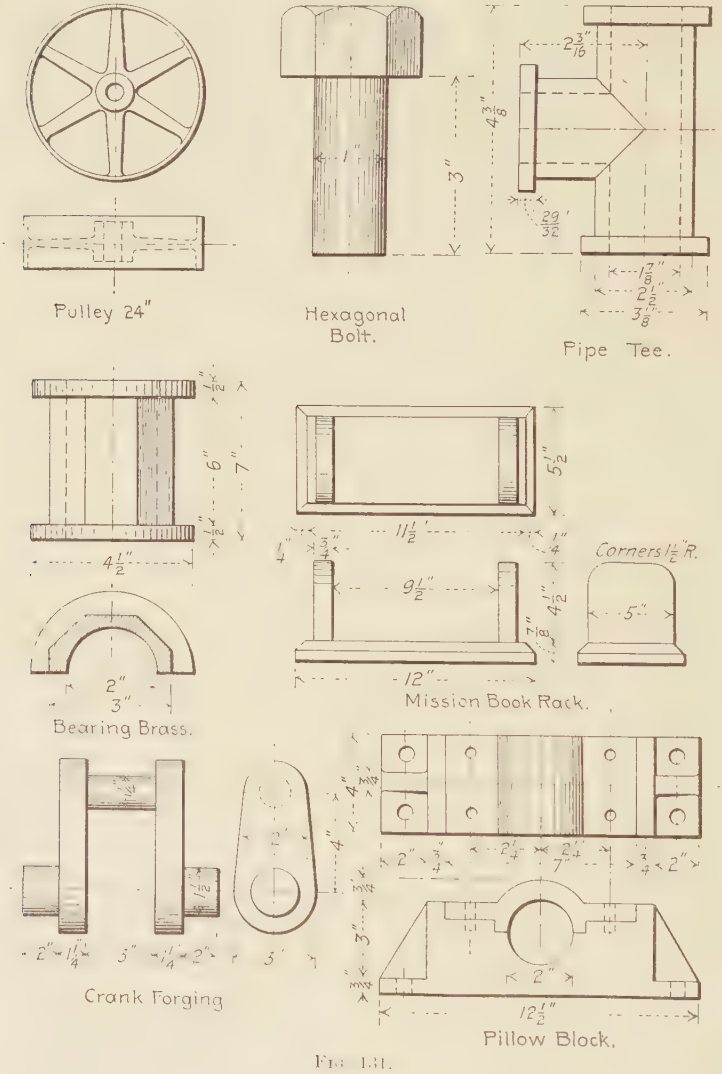


FIG. 131.

1. Circumscribe about each base a rectangle. The sides of these rectangles will be isometric lines.

2. Draw isometrically (Fig. 130*a*) the rectangle A′B′C′D′.

3. Locate the point O′ by ordinates.

4. Through the point O′ draw isometrically the rectangle E′F′G′H′.

5. Locate by ordinates the points 1, 2, 3, 4, 5, 6, 7, and 8.

6. Connect the proper points in order.

Note.—Do not draw in the invisible lines. They are drawn here to make the figure complete, but in practical work, where isometric drawings are used, the dotted lines are seldom drawn.

261. Other Methods of Pictorial Projection.

Under certain conditions, a closer approximation of a perspective than that shown by the isometric is obtained by the use of other oblique projections, cal'ed **Cavalier** and **Cabinet Projections.**

Cavalier Projection.—The object is placed with one face parallel to the picture plane, and the projectors are considered inclined 45° to the picture plane. Thus one face is the same as the orthographic view would be, and all lines perpendicular to the picture plane are projected in their true length. It thus becomes a form of isometric projection, with less distortion, because one face is the same as orthographic projection would make it.

Cabinet Projection.—To further reduce the distortion, the receding lines are shortened one-half. Otherwise the drawing is made like cavalier projection.

Good treatment of these and other styles of single plane projection, with numerous illustrations, is found in Prof. French's "Engineering Drawing."

EXERCISES IN ISOMETRIC DRAWING

262. Make isometric drawings of the following objects, to scale wherever necessary. One space to each.

851. Wooden box, without cover, 10 in. long, 6 in. wide, 4 in. deep, made of ¾-in. boards. Stand it on end, open to the front.

852. Draw the box in Ex. 851 in natural position with a cover, open 120°.

853. Pentagonal plinth, 2-in. diameter, ½ in. thick, surmounted by a 1-in. cylinder, 1½ in. high.

854. Bolt shown in Fig. 131.

855. Book rack in Fig. 131.

856. Pulley 24-in. diameter, 8-in. face, 2-in. bore (Fig. 131).

857. Bearing Brass for 2-in. shaft (Fig. 131).

858. Crank Forging, 4-in. throw (Fig. 131).

859. 1½-in. Pipe Tee (Fig. 131).

860. Pillow Block for 2-in. shaft (Fig. 131).

861. Piping Layout, shown in Fig. 132. This is intended for a full sheet, and may be drawn with a Title and Bill of Material. A tracing made of this exercise makes a splendid drawing.

Note.—First make the layout of the center lines, making allowances for all the connections.

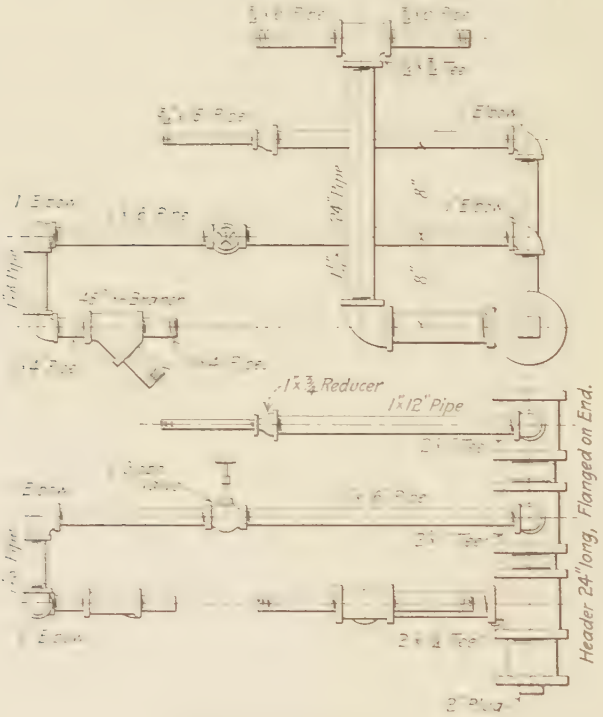


FIG. 132.—Piping layout

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